UNIVERSITY OF HAWAII
LIBRARY

ADVANCES IN PHYSICS

A QUARTERLY SUPPLEMENT
of the
PHILOSOPHICAL MAGAZINE

EDITOR
PROFESSOR N. F. MOTT, M.A., D.Sc., F.R.S.

EDITORIAL BOARD
SIR GEORGE THOMSON, M.A., D.Sc., F.R.S.
PROFESSOR A. M. TYNDALL, C.B.E., D.Sc., F.R.S.
SIR LAWRENCE BRAGG, O.B.E., M.C., M.A., D.Sc., F.R.S.

VOLUME 1 JULY 1952 NUMBER 3

PRICE per part 15*s.* 0*d.*
PRICE per annum £2 15*s.* 0*d.* post free

PRINTED AND PUBLISHED BY TAYLOR & FRANCIS LTD.
RED LION COURT, FLEET ST., LONDON E.C.4

QC1
A36

The Scientific Work of René Descartes

(1596–1650)

By

J. F. SCOTT, B.A., M.Sc., Ph.D.

With a foreword by H. W. TURNBULL, M.A., F.R.S.

This book puts the chief mathematical and physical discoveries of Descartes in an accessible form, and fills an outstanding gap upon the shelf devoted to the history of philosophy and science.

There is to be found in this volume the considerable contribution that Descartes made to the physical sciences, which involved much accurate work in geometrical optics and its bearing upon the practical problem of fashioning lenses, as also the deeper problems of light and sight and colour. The careful treatment that Dr. Scott has accorded to this work of Descartes is welcome, is well worth reading and will be an asset to all libraries. Publication is recommended and approved by the Publication Fund Committee of the University of London.

212 *pages*, 7″ × 10″, *amply illustrated.*

Price £1 - 0 - 0 net **Published July 1952**

Printed & Published by

TAYLOR & FRANCIS LTD.

RED LION COURT, FLEET STREET, LONDON, E.C.4

CONTENTS

THE MATHEMATICAL THEORY OF STATIONARY DISLOCATIONS

By F. R. N. NABARRO
Department of Metallurgy, The University of Birmingham

PART I. INTRODUCTION 271

PART II. DISLOCATIONS IN AN ELASTIC CONTINUUM . . . 272
§ 1. The Development of Dislocation Theory 272
§ 2. Some Elementary Properties of Dislocations . . . 277
§ 3. The Formal Elastic Theory of Dislocations . . . 287
§ 4. The Motion of Dislocations 295
§ 5. Arrays of Dislocations 311
§ 6. Interaction of Dislocations with Free Surfaces . . . 331
§ 7. The Line Energy of Dislocations 334

PART III. EFFECT OF THE CRYSTAL STRUCTURE . . . 348
§ 1. Elastic Dislocations and Crystal Dislocations . . . 348
§ 2. The Approximation of Peierls 350
§ 3. The Approximation of Frenkel and Kontorova . . 365
§ 4. Interaction of an Edge Dislocation with the Lattice Potential 369
§ 5. Dislocations in Special Crystal Structures . . . 376

ADVANCES IN PHYSICS

A QUARTERLY SUPPLEMENT

of the

PHILOSOPHICAL MAGAZINE

VOLUME 1 JULY 1952 NUMBER 3

The Mathematical Theory of Stationary Dislocations

By F. R. N. NABARRO
Department of Metallurgy, The University of Birmingham

CONTENTS

PART I. INTRODUCTION

PART II. DISLOCATIONS IN AN ELASTIC CONTINUUM

§ 1. THE DEVELOPMENT OF DISLOCATION THEORY.

§ 2. SOME ELEMENTARY PROPERTIES OF DISLOCATIONS.
  2.1. The Stress Field of an Edge Dislocation.
  2.2. The Screw Dislocation.
  2.3. A Dislocation cannot end Inside the Medium.

§ 3. THE FORMAL ELASTIC THEORY OF DISLOCATIONS.
  3.1. Weingarten's Theorem on the Possible Types of Dislocation.
  3.2. The Six Simple Types of Dislocation.
  3.3. More General Dislocations.
  3.4. Burgers's Theory of Line Dislocations.
  3.5. The Energies of Elastic Dislocations.

§ 4. THE MOTION OF DISLOCATIONS.
  4.1. Possible Motions of a Dislocation.
  4.2. The Frank–Read Source.
  4.3. The Force on a Dislocation : Law of Constant Resolved Shear Stress.
  4.4. Explicit Calculation of the Force on a Dislocation.
    4.41. Force on an Edge Dislocation.
    4.42. Force on a Screw Dislocation.
    4.43. Force Expanding an Infinitesimal Loop of Dislocation.
  4.5. Force on a Dislocation in a Varying Stress Field.
  4.6. Forces between Dislocations.
    4.61. Force between Edge Dislocations.
    4.62. Force between Screw Dislocations.
    4.63. Force between Parallel Dislocations with Burgers Vectors not Parallel.
    4.64. Forces between Dislocations along Perpendicular Lines.
    4.65. Dislocations in Thin Plates.
    4.66. Force between a Dislocation and a Solute Atom.

§ 5. ARRAYS OF DISLOCATIONS.
  5.1. A Row of Dislocations on the same Glide Plane.
  5.2. The Grain Boundary as a Wall of Dislocations.
    5.21. A Wall of Edge Dislocations.
    5.22. A Wall of Screw Dislocations.
    5.23. The General Grain Boundary as a Wall of Dislocations.
  5.3. Lattices of Dislocations.
    5.31. Taylor's Theory of Work Hardening.
    5.32. Influence of Dislocations on the Shear Modulus.

§ 6. INTERACTION OF DISLOCATIONS WITH FREE SURFACES.
  6.1. Elastic Interaction.
    6.11. The Screw Dislocation in a Circular Cylinder.
    6.12. The Edge Dislocation in a Circular Cylinder.
  6.2. Influence of the Surface Energy.
    6.21. Energy of an Exposed Ledge.
    6.22. Hollow Dislocations and Capillary Equilibrium.

§ 7. THE LINE ENERGY OF DISLOCATIONS.
  7.1. The Elastic Energy of a Circular Dislocation.
  7.2. Activation Energy Problems.
    7.21. The Models of Becker and Orowan.
    7.22. The Model of Mott and Nabarro.
    7.23. The Models of Orowan and of Frank and Read.
    7.24. The Model of Cottrell and Bilby.
    7.25. A Model of an Over-aged Alloy.
    7.26. A Model of a Localized Barrier.
  7.3. The Average Stress on a Dislocation which is nearly Straight.

PART III. EFFECT OF THE CRYSTAL STRUCTURE

§ 1. ELASTIC DISLOCATIONS AND CRYSTAL DISLOCATIONS.

§ 2. THE APPROXIMATION OF PEIERLS.
  2.1. The Solution for a Single Dislocation.
  2.2. The Solution for a Pair of Dislocations.
  2.3. Arrays of Dislocations.
    2.31. Edge Dislocations on the Same Glide Plane.
    2.32. Screw Dislocations Forming a Grain Boundary.
    2.33. Edge Dislocations Forming a Grain Boundary.
  2.4. Deviations from a Sinusoidal Law of Force.
    2.41. The Method of Foreman, Jaswon and Wood.
    2.42. The Structure and Energy of a Tilt Boundary.
  2.5. Barrier Problems.

§ 3. THE APPROXIMATION OF FRENKEL AND KONTOROVA.
  3.1. Introduction of a Second Harmonic.
  3.2. Perturbation Theory.

§ 4. INTERACTION OF AN EDGE DISLOCATION WITH THE LATTICE POTENTIAL.
  4.1. Change in Energy of a Straight Edge Dislocation as it moves through the Lattice.
  4.2. The Zig-zag Dislocation.

§ 5. DISLOCATIONS IN SPECIAL CRYSTAL STRUCTURES.
  5.1. Reactions and Nodes.
  5.2. Partial Dislocations.
    5.21. Mobile Partial Dislocations.
    5.22. Sessile Dislocations.
  5.3. The Choice of a Glide System.
    5.31. Special Lattices.
  5.4. Twinning and Phase Transformations.
    5.41. Twin Lattices, Twinning and Pole Dislocations.
    5.42. Special Lattices.
    5.43. The Analysis of Lifshits and Obreimov.

REFERENCES.

PART I. INTRODUCTION

DISLOCATIONS were first introduced into physical theory as part of the study of a mechanical aether. Burton (1892) postulated that ' a given portion of matter consists, not of any individual portion of aethereal or other substance, but of modifications in the structure . . . . of the aether, and when matter moves it is merely these modifications of structure . . . . which are transferred from one portion of the aether to another '. As a typical modification he considered a ' strain-figure '. He realized that these figures could only persist in a medium with non-linear elastic properties: ' Let the medium . . . . be held in its strained condition by some compelling agency: . . . . provided the strain has at no point too great a value, the original condition will be completely regained after the compelling agency has been removed. But suppose . . . . that the restoring stresses do not always increase with the strain, but that beyond a certain point in the process they begin to fall off in value, until at last a point is reached at which the general tendency of the stresses is to further increase the strain. If the compelling agency is now withdrawn, the medium will subside into a new condition of stable equilibrium, involving stress and strain at every point. The state of things . . . . will be referred to as a *strain-figure*, and we may now proceed to examine its dynamical properties '. These include the ability to travel freely with any velocity small compared with the speed at which disturbances of the same type as the strains in the strain-figure are propagated through the medium, the possession of an ' effective mass ', and the existence of forces between strain-figures. ' The *coarse-grainedness* of the medium has an influence in determining the *size* of possible strain-figures.' Larmor (1897) used a similar model, considering as examples a wire ' welded into a ring after twist has been put into it ', and a ' breach of continuity ' produced ' by the creation of a lens-shaped cavity, and the material on one side of the breach is twisted round in its plane, and continuity is then restored by cementing the two sides together '. He pointed out later (1900) that ' It is the intrinsic strain-form alone that constitutes the electron; and it is a fundamental postulate that the form can move from one portion to another of the stagnant aether somewhat after the manner that a knot can slip along a cord. If this form is taken as an entity, so that its position is part of the specification of the system, then the dynamical analysis introduces forces acting on it. . . . When an intrinsic strain-form is implanted in an elastic solid, ... it cannot in general be removed to another position even by imposing additional strains in the medium, because the discontinuities involved in its creation cannot be entirely annulled by imposing any continuous strain: thus there cannot be phenomena of freely mobile strain-forms connected with a solid medium '.

Larmor's theory of strain-figures in a medium showing strictly linear elasticity, except at some time in the past when violent cutting and welding operations were permitted, has developed into the theory of elastic dislocations described in Part II of this article; Burton's theory of strain-figures in a granular medium which does not obey Hooke's law for large strains corresponds to the theory of dislocations in a crystal lattice described in Part III. In this article we deal only with stationary dislocations, and we do not discuss the influence of dislocations on the scattering of x-rays (Wilson 1949, Frank 1949 a) or on the electrical (Koehler 1949, Mackenzie and Sondheimer 1950, Dexter 1952) and magnetic (Brown 1941) properties of the material. Nor do we describe the important role of dislocations in crystal growth, of which accounts have recently been given by Burton, Cabrera and Frank (1951), Burton (1951) and Frank (1952 a).

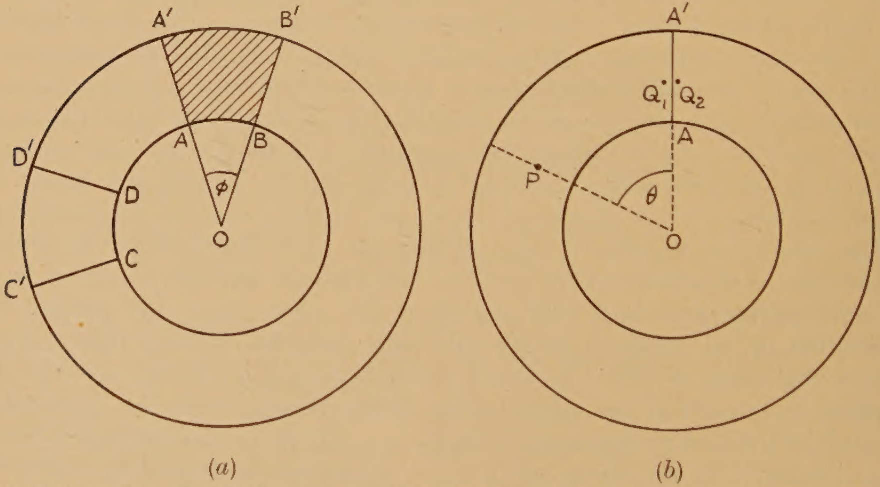
This article was originally written during my tenure of a Royal Society Warren Research Fellowship at the University of Bristol. It has been considerably improved by the criticisms of my colleagues in Bristol and afterwards in Birmingham: in particular, I should like to mention the continual help of Dr. J. D. Eshelby.

PART II. DISLOCATIONS IN AN ELASTIC CONTINUUM

§1. THE DEVELOPMENT OF DISLOCATION THEORY

The ordinary theory of elasticity requires that the displacement at any point of a body should be a single valued function of the coordinates of that point. This restriction disappears if the body is multiply connected. For example, the ring of fig. 1 (*a*) may be cut along the radial lines OAA′, OBB′, and the wedge AA′B′B of angle ϕ removed.

Fig. 1



(*a*) (*b*)

(*a*) Dislocation of a hollow ring. The ring is cut along the radial lines AA′ and BB′, the wedge AA′B′B is removed, and AA′ is joined to BB′. The same dislocation could be made by cutting along CC′ and DD′. (*b*) The dislocated ring. The neighbouring points Q_1 and Q_2 lie on opposite sides of the cut.

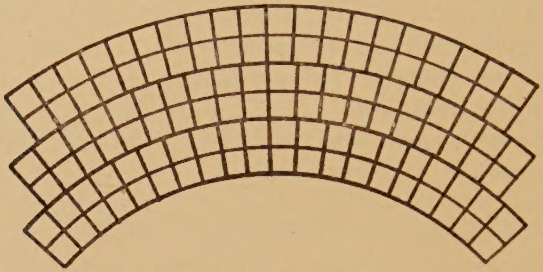
If the cut edges are now joined, the ring is in a state of self-strain. The coordinates of a point P in the strained ring may be expressed in polar coordinates r=OP, θ=angle AOP (fig. 1 (*b*)). The displacement u_θ of the point P is given by $u_\theta = r\theta\phi/2\pi$. For the point Q_1 the displacement is zero; for the neighbouring point Q_2 the displacement is $r\phi$. We may either put a mathematical barrier along AA′, and say that the displacement is discontinuous across the barrier, or else say that the displacement is continuous, but not single valued. The possibility of doing this rests on the fact that the physical state of any part of the body does not depend directly on the components of displacement, but only on their derivatives. The addition of a displacement corresponding to that of a rigid body does not alter the strains, and the difference of any two of the multiple values of the strain is a rigid-body displacement. If we use continuous multiple valued displacements, the component u_θ at Q_1 has the values $0, r\phi, 2r\phi, \ldots, -r\phi, -2r\phi, \ldots$, and the second of these is continuous with the displacement of Q_2. Either of these methods of approach is mathematically satisfactory, but the continuous displacement gives a better physical picture, for there is nothing in the ring of fig. 1 (*b*) to distinguish the line AA′ from any other radial line. Instead of cutting out the wedge AA′B′B, we might have cut out the equal wedge CC′D′D, and the resulting self-strained ring would have been identical with that of fig. 1 (*b*).

Displacements of this kind in isotropic elastic bodies were studied by the Italian school of mathematicians under the name of ‘ distorsioni ’. The principal results were summarized by Volterra (1907), and more recent accounts have been given by Love (1927), Neményi (1931) and Goodier (1939). Love ‘ ventured to call them “ dislocations ” ’, and the word has been generally adopted.

The first application of dislocations to the theory of plastic deformation was made by Prandtl in a lecture course in 1921–2, in which he developed a model to explain the plastic properties of crystals by assuming that in regions of imperfection there were molecules which could change their allegiance from one relatively perfect lattice to another. Prandtl (1928) did not indicate the connection of these imperfections with the dislocations of the theory of elasticity, although the mechanical model he devised to represent his ideas contains one idea that was essential to the further development of a theory of crystal slip. The solid is imagined divided into upper and lower parts. The periodic field of the lower part is represented by a corrugated surface, while a particular atom of the upper part is represented by a heavy roller which can move across these corrugations. The roller is connected by a system of springs and weights to a rigid framework representing the rest of the upper part. A shear of the solid is represented by moving the rigid frame in a direction perpendicular to the corrugations. For small strains the roller moves reversibly in one trough; for larger strains the roller jumps irreversibly into another trough. If the shift of the rigid framework is equal to the wavelength of the corrugations, and the roller jumps to the next trough, the forces acting on the roller return to their original values. The process may be repeated from trough to trough.

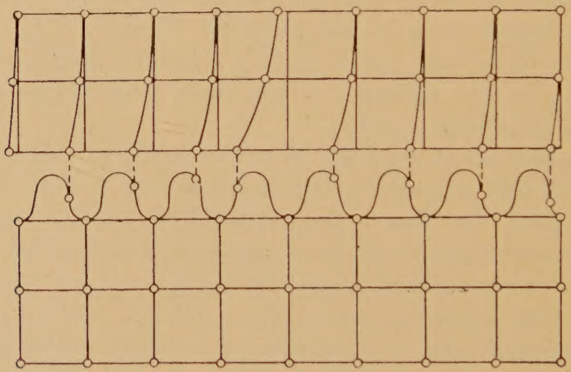
In considering the plastic deformation of single crystals, Masing and Polanyi (1923) were led to consider the imperfections which must exist in a bent crystal. They suggested a type of deformation ('Biegegleitung') in which sheets of crystal bent elastically were joined through surfaces of misfit as in fig. 2. The lattices on either side of each surface of misfit are uniformly expanded or compressed, and atoms above and below the surface are out of register over almost the whole surface. It follows from the work of van der Merwe (§ 2.31 of Part III) that a structure of this kind would not be stable. A closer approach to the modern idea of a dislocation was provided by the imperfection which Dehlinger (1929) introduced under the name of 'Verhakung' (fig. 3). In this, the disturbance of the lattice is confined to a small region of the surface of

Fig. 2



'Biegegleitung' as envisaged by Masing and Polanyi. The crystal breaks up into a pile of elastically bent lamellae.

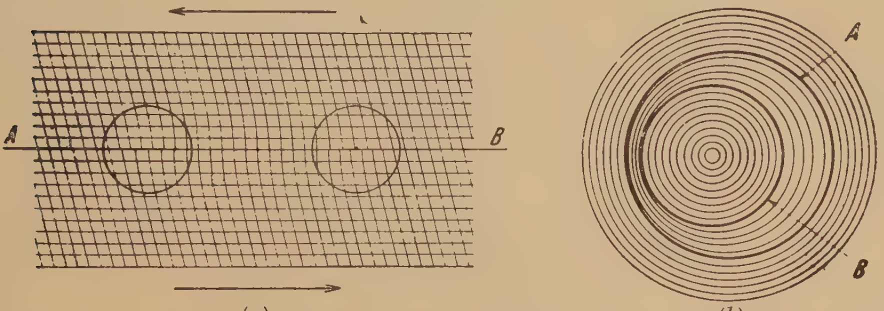
Fig. 3



Dehlinger's picture of a 'Verhakung'. Six rows of atoms in a simple square lattice. The atoms in the three upper rows are displaced : the sinusoidal curve shows the potential in which the atoms in the third row move on account of their interaction with the atoms in the fourth row.

misfit, and outside this region the atoms above and below the surface are in almost perfect register. On Dehlinger's rough mathematical approximations the localization of the disturbance is even more marked than in a more exact treatment. His ' Verhakung ' in fact represents a pair of dislocations of opposite sign, separated by only one lattice spacing. Dehlinger's idea was important in directing attention to localized imperfections, and in indicating that the lattice planes perpendicular to the glide plane must become curved in the neighbourhood of the dislocation.

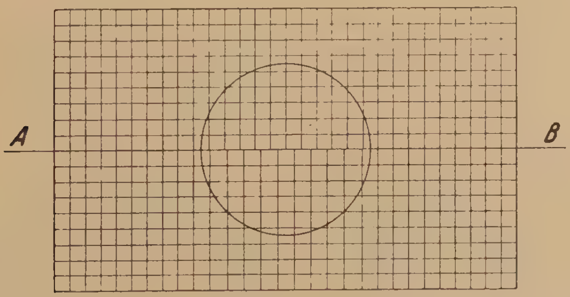
Fig. 4



(*a*) (*b*)

Orowan's model of a local gliding. (*a*) Section in the glide direction perpendicular to the glide plane AB. The lattice planes were originally straight and orthogonal. The rings indicate the positions of the dislocations. (*b*) Schematic view in the glide plane. The circles were originally concentric ; the dislocation lies between the circles A and B.

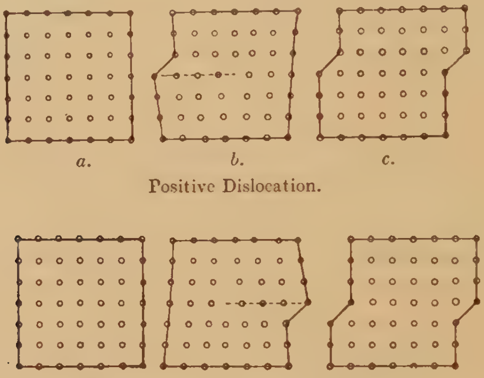
Fig. 5



Polanyi's picture of a dislocation. The glide plane is AB.

The foundations of the modern dislocation theory of slip were laid by Orowan, Polanyi and Taylor in 1934. Orowan (1934 b) showed (fig. 4) that if a limited area of one plane slips by one atomic distance over the neighbouring plane, the boundary of this area is a closed loop of dislocation. Once this loop is formed, glide can propagate across the plane by the spreading of the area, which is a motion of the dislocation line across its glide plane. Polanyi (1934) sketched the form (fig. 5) of a single dislocation at rest in an unstrained lattice, and showed that it could move across the glide plane under stresses much less than that required to slide one lattice plane rigidly across the other. Polanyi's dislocation is confined to a small region of the glide plane, but the strains within the region of the dislocation are still uniform, as they are in fig. 2. The planes perpendicular to the glide plane are not bent, and the separation of pairs of planes passing near the dislocation does not tend to the unstrained value at large distances. Polanyi estimates that if the dislocation is n atoms wide, the shear stress required to move it is $1/n$ of the shear strength of a perfect lattice. This would require n to be at least 1 000 in a soft single crystal, which seems improbably high. The estimate of §4.1 of Part III leads to a much smaller stress. If the dislocation moves across the crystal from one side to the other, the material above the glide plane slips one atomic distance over that below.

Fig. 6



Taylor's model of the atomic positions before, during and after the passage of a dislocation across the crystal.

Taylor (1934), like Polanyi, considered dislocations lying along straight lines in the glide plane perpendicular to the glide direction. He pointed out the connection of these dislocations with the elastic dislocations of Volterra, calculated (though with an error) the stress field surrounding such a dislocation, and studied the interaction between dislocations. His sketch (fig. 6) of the positions of atoms surrounding a dislocation showed for the first time that nearly all the lattice planes perpendicular to the glide plane remain continuous, though they are curved.

Burgers (1939 a), considering the connection between Taylor's two dimensional lattice dislocation and the elastic dislocations of Volterra, was led to consider 'screw' dislocations, which lie along lines in the glide plane parallel to the direction of slip, and Frank (1948, 1949 b) has emphasized the importance of this type of dislocation. Since then the theory has been extended by a number of workers, but it is only recently that attempts have been made to relate dislocation theory to the actual crystal structures of individual metals (see Mott 1951, 1952a). Surveys of the dislocation theory of slip have been given by Cottrell (1949) and Seitz (1950 a).

§2. SOME ELEMENTARY PROPERTIES OF DISLOCATIONS

2.1. THE STRESS FIELD OF AN EDGE DISLOCATION

Consider an isotropic medium having rigidity μ and Poisson's ratio σ. In this medium lies a dislocation of the kind sketched in figs. 5 and 6, which we shall call an 'edge' dislocation. Conflicting formulae have been given for the stress distribution around such a dislocation: the present analysis is based on that of Koehler (1941). We take the Airy stress function

$$\chi = -\tfrac{1}{2} D y \ln b^{-2} (x^2+y^2) + Bx \tan^{-1} (y/x), \quad . \quad . \quad . \quad . \quad (1)$$

in terms of which the components of stress are given by

$$\left.\begin{aligned} p_{xx} &= \frac{\partial^2 \chi}{\partial y^2} \\ p_{xy} &= -\frac{\partial^2 \chi}{\partial x \partial y} \\ p_{yy} &= \frac{\partial^2 \chi}{\partial x^2}. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

To obtain Taylor's expressions for the stresses we take

$$D = -B = b\mu/\pi, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

while for J. M. Burgers's expressions we take

$$D = b\mu/2\pi(1-\sigma), \quad B = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In these formulae b is the strength of the dislocation, equal to the relative displacement of the two parts of a crystal when the dislocation crosses the glide plane. If the stresses are calculated from (1), (2) and (3), it is easy to see that the y component of the resultant force transmitted across a circular boundary centred at the origin does not vanish, but has the value $2\pi B = -2b\mu$ for each unit length of the dislocation. This stress system can only be maintained by a force of $-2b\mu$ on the boundary at infinity, and an equal and opposite force applied at the origin. Instead of calculating the stresses and integrating, we may use the theorem of Michell (1899) that the resultant forces acting on the boundary are given by

$$\left.\begin{aligned} x &= -\int \frac{d}{ds}\left(\frac{\partial \chi}{\partial y}\right) ds \\ y &= \int \frac{d}{ds}\left(\frac{\partial \chi}{\partial x}\right) ds, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where the integrals are taken round any closed circuit containing the dislocation. Single valued terms in $\partial\chi/\partial y$ and $\partial\chi/\partial x$ do not contribute to these integrals: only the term $B\tan^{-1}(y/x)$ in $\partial\chi/\partial x$ contributes a residue of $2\pi B$.

Both these forces at the origin and the corresponding couple, which depends (Filon 1921) on an integral of the type of (5) involving χ, vanish for Burgers's stress function, which thus represents an unconstrained dislocation in an isotropic medium.

The displacements are given (Love, p. 205) by

$$\left.\begin{aligned} 2\mu u_x &= -\frac{\partial\chi}{\partial x}+(1-\sigma)\frac{\partial\psi}{\partial y} \\ 2\mu u_y &= -\frac{\partial\chi}{\partial y}+(1-\sigma)\frac{\partial\psi}{\partial x}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where

$$\frac{\partial^2\psi}{\partial x\partial y}=\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0.$$

For Burgers's stress function

$$\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2}=-\frac{2Dy}{x^2+y^2},$$

and the function $\psi=-D[x\ln b^{-2}(x^2+y^2)-2x+2y\tan^{-1}(x/y)]$ satisfies (7). The corresponding displacements are

$$\left.\begin{aligned} u_x &= \frac{b}{4\pi(1-\sigma)}\frac{xy}{x^2+y^2}-\frac{b}{2\pi}\tan^{-1}\frac{x}{y} \\ u_y &= -\frac{(1-2\sigma)b}{8\pi(1-\sigma)}\ln b^{-2}(x^2+y^2)+\frac{b}{4\pi(1-\sigma)}\frac{y^2}{x^2+y^2}. \end{aligned}\right\} \quad . \quad (8)$$

The stresses are given by

$$\left.\begin{aligned} p_{xx} &= -Dy(3x^2+y^2)/(x^2+y^2)^2 \\ p_{xy} &= Dx(x^2-y^2)/(x^2+y^2)^2 \\ p_{yy} &= Dy(x^2-y^2)/(x^2+y^2)^2. \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (9)$$

The equivalent expressions in polar coordinates are given in § 4.41.

These formulae, which are given by Burgers, are equivalent to those of Koehler, since the origin of u_x and u_y is arbitrary. It appears that u_y is logarithmically divergent at large distances from the dislocation. The strain and stress correspondingly tend to zero only as fast as r^{-1} at large distances r, and the energy of the strained medium is not localized near the origin, but diverges logarithmically at infinity. Because of this divergence, the mathematical theory of dislocations presents unexpected difficulties.

The displacement u_y is single valued, but u_x has a residue of b for every circuit enclosing the origin, corresponding to the fact that every circuit of this kind cuts once the surface across which the lattice has slipped by a distance b in the x direction.

There are no other two dimensional displacements which have the essential properties of a dislocation. These properties are (a) that one component of displacement (which we take to be u_x) should have the same residue for *any* large circuit enclosing the dislocation, because such a circuit cuts the line over which slip has occurred once, (b) that the other component u_y should be single valued, (c) that no resultant force or moment acts across a circuit surrounding the dislocation, and (d) that the energy should be as small as possible. All of these conditions, except possibly (d), are satisfied by the Burgers dislocation. If we work in polar coordinates, the stress function is of the form $\chi=-Dr \ln r \sin\theta$, and the stresses are everywhere of order $1/r$. Suppose another dislocation with the same strength b, but less energy, can be made by adding to χ another stress function χ^1. Then stresses corresponding to χ^1 must also be of order $1/r$ at large distances. For if the stresses tended to zero more slowly, the self energy of χ^1 would diverge at infinity more rapidly than the interaction energy of χ and χ^1, and the addition of χ^1 would increase the total energy. If the stresses tended to zero more rapidly, the interaction energy would be negligible in comparison with the self energy of χ. We therefore require that each of the expressions

$$\left.\begin{aligned} p_{rr}&=\frac{1}{r}\frac{\partial\chi^1}{\partial r}+\frac{1}{r^2}\frac{\partial^2\chi^1}{\partial\theta^2}\\ p_{r\theta}&=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi^1}{\partial\theta}\right)\\ p_{\theta\theta}&=\frac{\partial^2\chi^1}{\partial r^2}\end{aligned}\right\} \quad \ldots\ldots \quad (10)$$

should be of the form $r^{-1}a_{ij}(\theta)$ when r is large. This requires that χ^1 should be of the form

$$\chi^1=Fr\ln r\cos\theta+Gr\ln r\sin\theta+rH(\theta),$$

with stresses

$$\begin{aligned} p_{rr}&=Fr^{-1}\cos\theta+Gr^{-1}\sin\theta+r^{-1}[H''(\theta)+H(\theta)]\\ p_{r\theta}&=Fr^{-1}\sin\theta-Gr^{-1}\cos\theta\\ p_{\theta\theta}&=Fr^{-1}\cos\theta+Gr^{-1}\sin\theta.\end{aligned}$$

The term in G is of the same form as χ, while that in F is the same deformation turned through a right angle, and so violates condition (b). To determine the form of $H(\theta)$, we make a Fourier analysis of $H''(\theta)+H(\theta)$. All terms in this analysis contribute to the self-energy of χ^1, while the only term which can reduce the energy by its interaction with G is the term in $\sin\theta$. Thus $H''(\theta)+H(\theta)=\sin\theta$, which gives

$$H(\theta)=-\tfrac{1}{2}\theta\cos\theta+F^1\cos\theta+G^1\sin\theta.$$

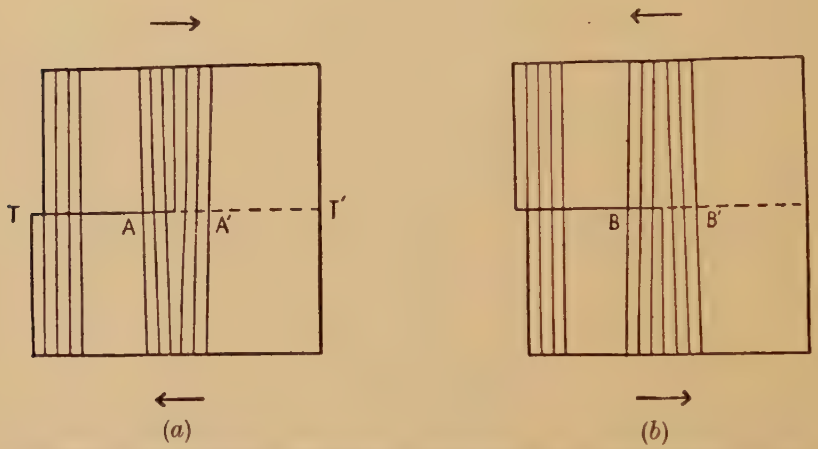
The terms in F^1 and G^1 lead to zero stresses, and there remains the term $\chi^1=-\frac{1}{2}r\theta\cos\theta$. This is the term with coefficient B in eqn. (1), which has already been rejected because it is not in equilibrium unless a force is applied at the origin. These three terms, together with $H(\theta)=\frac{1}{2}\theta\sin\theta$, are the only biharmonic functions of the form $rH(\theta)$.

László (1950) has suggested that the two dimensional dislocation discussed here may not be the form of lowest energy, since it contains a long cylinder of compressed region which might reduce its energy by buckling, while the energy of the surrounding material increases. The problem has not been studied quantitatively.

2.2. THE SCREW DISLOCATION

Figure 7 represents schematically the positive and negative dislocations of fig. 6. In fig. 7 (*a*) glide has started at T on one side of the crystal, and has not yet reached the other end T′ of the slip plane. There is a region AA′ of the slip plane which contains an edge dislocation in which n atoms above the plane are associated with $n-1$ atoms below. In the ideal two dimensional edge dislocation of § 2.1, the same disturbance appears in all sections parallel to the plane of fig. 7. We must expect that in a real crystal this ideal two dimensional picture will not be valid, and that the region of slip TA will have progressed further across the slip plane in some sections than in others. Looking down on the glide plane of fig. 7, we shall see the dislocation schematically as in fig. 8. The parts of the dislocation represented by single lines are edge dislocations such as we have already considered. The parts represented by broken lines are screw dislocations of the type introduced into plasticity theory by Burgers. Whereas the line of an edge dislocation

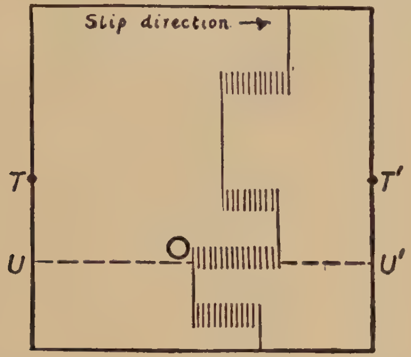
Fig. 7



(*a*) Passage of a positive dislocation across the glide plane from T to T′.
(*b*) A negative dislocation.

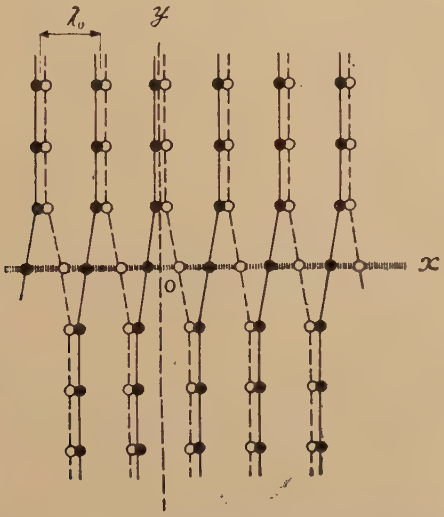
is perpendicular to the slip direction, a screw dislocation is a line parallel to the slip direction which separates regions of the glide plane over which slip has already occurred from regions which have not yet slipped.

Fig. 8



The glide plane of the crystal of fig. 7. The portion of crystal to the left of the boundary has slipped, while that to the right has not. Continuous lines in the boundary are segments of edge type, broken lines are segments of screw type dislocation.

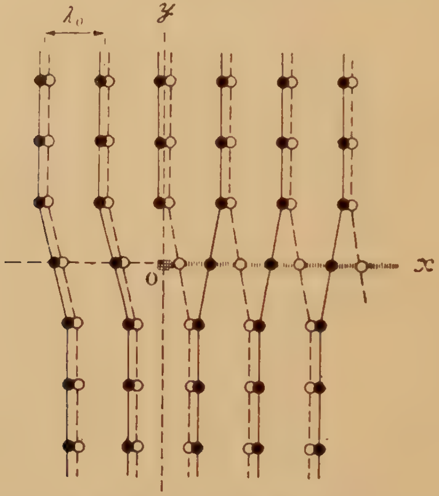
Fig. 9



A screw dislocation along the x axis (after J. M. Burgers). Continuous lines indicate rows of atoms above the xy plane, dotted lines indicate rows below this plane.

Figure 9 shows the displacement of atoms above and below the slip plane in the neighbourhood of a screw dislocation. Although we have associated a screw dislocation with an edge dislocation lying in a particular glide plane, the screw dislocation can move in any plane passing through its axis, and its sections by all of these planes are similar. For example, figs. 8 and 9 show the section of a screw dislocation by a plane passing through TT′ of figs. 7 and 8, and perpendicular to the plane of fig. 7. Suppose we now take a section parallel to that of fig. 7, and passing, not through TT′ of fig. 8, but through UOU′. Then (fig. 10) the section OU′ of the screw dislocation appears as in fig. 9. At O we have slight distortion of the lattice without any dislocation below the plane of the figure, and an edge dislocation in planes above that of the figure. This is an example of the general rule that a dislocation cannot end in the crystal. Its segments of edge, screw or mixed type must extend continuously from one boundary to the other, or else form closed loops (cf. § 2.3). By a segment of mixed type is meant one which lies neither perpendicular nor parallel to the slip direction. Introducing these, we see that a dislocation need not form a rectangular zig-zag as in fig. 8, but may follow any smooth curve in the glide plane. The loop dislocation of fig. 4 is an example of a dislocation composed of segments of mixed type.

Fig. 10



The same dislocation as in fig. 9 lies along the positive x axis. At the origin the dislocation line bends into the direction Oz; the plane of atoms above the xy plane contains an edge dislocation (after J. M. Burgers).

The screw dislocation is less easy to visualize than the edge dislocation. It is perhaps most easily pictured by considering the ring of fig. 1 (*a*) to be a section of a long hollow cylinder. If this cylinder is cut by a plane such as C′C passing through its axis, and the surface above C′C slides over the surface below through one atomic distance in a direction parallel to the axis of the cylinder, the two surfaces may again be joined with almost perfect matching of the atomic lattices. If the same process is tried with a solid cylinder, there is no difficulty in the neighbourhood of C′, but there is a singularity with infinite elastic strain at the centre O. This is a screw dislocation.

Partly because of the difficulty of visualizing a three dimensional deformation, and partly because the original papers of Taylor and Polanyi considered only edge dislocations, little consideration was given to the effects of screw dislocations until about 1948. In fact, the screw dislocation, which can travel in any plane in which it lies, plays at least as important a part in plasticity as the edge dislocation, which can travel only in the direction of the displacement it produces.

The mathematical theory of screw dislocations is simpler than that of edge dislocations, for only pure shear strains are involved, without dilatation. For a screw dislocation lying along the *Oz* axis, the only component of displacement is u_z, and the displacement depends only on x and y, being given by

$$u_z=\frac{b}{2\pi}\tan^{-1}\frac{y}{x}. \qquad (11)$$

We use tensorial components of strain e_{ij} defined by

$$2e_{ij}=\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}, \qquad (12)$$

and only components of strain which do not vanish are

$$\left.\begin{aligned} e_{xz}&=-\frac{b}{4\pi}\frac{y}{x^2+y^2} \\ \text{and} \qquad e_{yz}&=\frac{b}{4\pi}\frac{x}{x^2+y^2}. \end{aligned}\right\} \qquad (13)$$

The corresponding stresses are

$$\left.\begin{aligned} p_{xz}&=-\frac{b\mu}{2\pi}\frac{y}{x^2+y^2} \\ p_{yz}&=\frac{b\mu}{2\pi}\frac{x}{x^2+y^2}. \end{aligned}\right\} \qquad (14)$$

In cylindrical coordinates these become

$$\left.\begin{aligned} u_z&=b\theta/2\pi \\ e_{\theta z}&=b/4\pi r \\ p_{\theta z}&=b\mu/2\pi r, \end{aligned}\right\} \qquad (15)$$

and all other components vanish. It will be seen that in this elastic solution the displacement is *independent of r*.

As in the case of edge dislocations, we may show that of all displacements which have only u_z components, and which depend only on x and y, only that given by (11) has the properties of a dislocation. For suppose that

$$u_x=u_y=0$$
$$u_z=U(x, y).$$

Then the only stresses are

$$p_{xz}=\mu\partial U/\partial x$$

and

$$p_{yz}=\mu\partial U/\partial y.$$

The equations of equilibrium in the x and y directions are automatically satisfied, and the condition for equilibrium in the z direction is

$$\frac{\partial p_{xz}}{\partial x}+\frac{\partial p_{yz}}{\partial y}=0$$

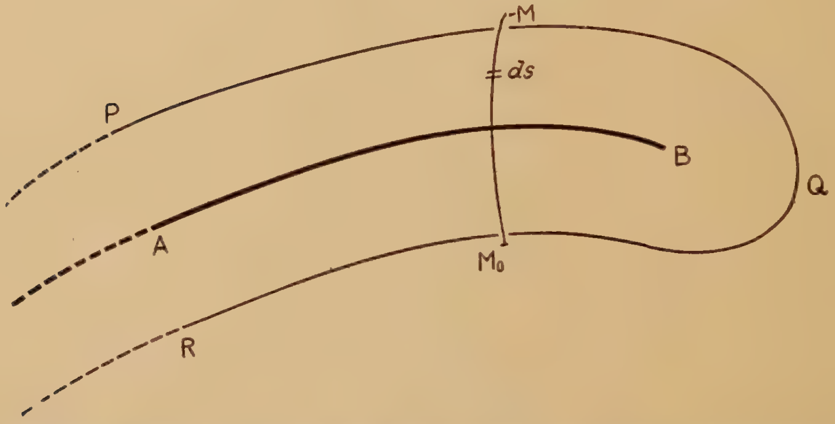
or

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0.$$

In cylindrical coordinates,

$$\left.\begin{aligned}p_{rz}&=\mu\partial U/\partial r\\ p_{\theta z}&=\mu\partial U/r\partial\theta,\end{aligned}\right\}\quad\ldots\ldots\ldots\quad(16)$$

Fig. 11



A dislocation line AB which ends in the medium at B may be surrounded by a cap PQR. Outside this cap the strain is continuous and can be differentiated at least twice. The circuit M_0M encircles the cap.

and, as in § 2.1, we need only consider harmonic functions U for which the stresses are proportional to $1/r$ at large distances. The only function of this kind is $E+F\ln r+G\theta$. The constant E leads to zero stress, and $U=G\theta$ is the screw dislocation just discussed. $F\ln r$ gives a single valued displacement which does not interact with $G\theta$, and a combination of these terms has a higher energy than the term $G\theta$ alone.

2.3. A DISLOCATION CANNOT END INSIDE THE MEDIUM

We shall first give a formal proof of this theorem by means of the theory of elasticity, and then illustrate its meaning by considering a dislocation in a crystal lattice. Suppose AB in fig. 11 represents a dislocation which ends in the medium at B. We take (cf. Love, p. 222) a point M_0 close to the line of the dislocation, write down the components u_i of the displacement, and evaluate the integral

$$\int \frac{du_i}{ds} ds, \quad \ldots \ldots \ldots \quad (17)$$

for a closed circuit starting at M_0 and encircling AB. Then if AB is a dislocation, the integral (17) does not vanish, and its value is the strength of the dislocation, a vector **b**. Since the line of the dislocation is likely to be a singularity of the strain, we surround it with a surface PQR and take the circuit outside this surface, where the strain is continuous and can be differentiated at least twice. Though the displacement may not be single valued, the strain, which determines the physical state of the medium, must be so.

Then

$$\begin{aligned} \frac{du_i}{ds} ds &= \frac{\partial u_i}{\partial x_j} dx_j \\ &= \left[\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)\right] dx_j \\ &= \left[e_{ij}-\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}-\frac{\partial u_i}{\partial x_j}\right)\right] dx_j. \quad \ldots \ldots \quad (18) \end{aligned}$$

The repeated suffix j in these formulae implies a summation over all values of j. Integrating from M_0 to an arbitrary point M on the circuit, we obtain

$$\begin{aligned} \int_{M_0}^{M} \frac{du_i}{ds} ds &= \int_{M_0}^{M} e_{ij}\, dx_j - \tfrac{1}{2} \int_{M_0}^{M} \left(\frac{\partial u_j}{\partial x_i}-\frac{\partial u_i}{\partial x_j}\right) d\,\{x_j - x_j(M)\} \\ &= \int_{M_0}^{M} e_{ij}\, dx_j - \tfrac{1}{2}\{x_j(M)-x_j(M_0)\}\left\{\frac{\partial u_j}{\partial x_i}(M_0)-\frac{\partial u_i}{\partial x_j}(M_0)\right\} \\ &\quad + \tfrac{1}{2}\int_{M_0}^{M} \{x_j - x_j(M)\}\frac{\partial}{\partial x_k}\left(\frac{\partial u_j}{\partial x_i}-\frac{\partial u_i}{\partial x_j}\right) dx_k. \quad \ldots \quad (19) \end{aligned}$$

If $\left(\frac{\partial u_j}{\partial x_i}-\frac{\partial u_i}{\partial x_j}\right)$ has more than one value, any one of these values may be selected, and inserted in both terms of (19), so that we follow the same branch of the many valued displacement functions.

Now

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial x_k}\left(\frac{\partial u_j}{\partial x_i}-\frac{\partial u_i}{\partial x_j}\right) &= \frac{1}{2}\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_k}+\frac{\partial u_k}{\partial x_j}\right)-\frac{1}{2}\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right) \\ &= \frac{\partial e_{jk}}{\partial x_i}-\frac{\partial e_{ki}}{\partial x_j}, \quad \ldots \ldots \ldots \quad (20) \end{aligned}$$

and so, changing the dummy index in the first integral on the right of eqn. (19) from j to k, we have

$$\int_{M_0}^{M} \frac{\partial u_i}{ds}\,ds = -\tfrac{1}{2}\{x_j(M)-x_j(M_0)\}\left\{\frac{\partial u_j}{\partial x_i}(M_0)-\frac{\partial u_i}{\partial x_j}(M_0)\right\}$$

$$+\int_{M_0}^{M}\left[e_{ik}+\{x_j-x_j(M)\}\left(\frac{\partial e_{jk}}{\partial x_i}-\frac{\partial e_{ki}}{\partial x_j}\right)\right]dx_k . \quad . \quad . \quad (21)$$

If the circuit is closed, so that M coincides with M_0, the first term on the right of (21) vanishes, while the second may be written $\int \xi_{ik}\,dx_k$,

where $$\xi_{ik}=e_{ik}+\{x_j-x_j(M_0)\}\left(\frac{\partial e_{jk}}{\partial x_i}-\frac{\partial e_{ki}}{\partial x_j}\right). \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Since the dislocation ends within the medium, we may replace the integral of ξ_{ik} round the circuit by the integral of its curl X_{ik} over a surface which has the circuit for an edge, and lies entirely within the medium and outside the surface PQR. The value of X_{ik} is given by

$$X_{ik}=\epsilon_{klm}\frac{\partial \xi_{im}}{\partial x_l}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where ϵ_{klm} is $+1$ if (klm) is an even permutation of (123), -1 if (klm) is an odd permutation of (123), and zero otherwise. If this expression is evaluated, it is found that X_{ik} vanishes on account of the conditions of compatibility of elastic strains. Thus (21) vanishes for a closed circuit, and the strength of the dislocation is zero.

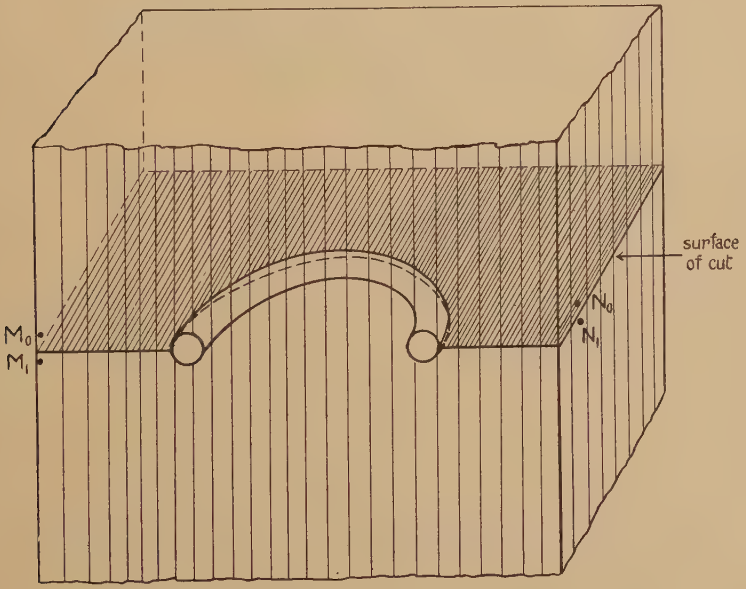
To visualize what happens if we try to end a dislocation in a crystal lattice, consider the dislocated plane of fig. 6 (*b*). Take any closed circuit surrounding the dislocation once anticlockwise, and count the number of lattice spacings up, down, left and right in the circuit. The number of steps up is equal to the number of steps down, but the number left always exceeds the number right by unity. This residue, which is analogous to the integral (17), is a measure of the strength of the dislocation. Now consider a large circuit of this kind, far from the dislocation, so that the atomic spacings on this circuit are nearly uniform. Build up the next sheet of atoms by putting in the atoms which lie immediately above this circuit and then filling in the closed ring of atoms as well as possible. Usually, the atoms in this closed ring will lie in one-to-one correspondence with the atoms of the circuit below. In this case the excess of steps left over steps right is unity for the new circuit also, and the new plane must contain a dislocation within the circuit. If there is not a one-to-one correspondence, a line of dislocation must cross from inside the boundary to outside at the point where this one-to-one correspondence breaks down. In either case the theorem is verified.

§3. The Formal Elastic Theory of Dislocations

3.1. WEINGARTEN'S THEOREM ON THE POSSIBLE TYPES OF DISLOCATION

We consider multiply connected bodies of two types. The first is exemplified by a torus, such as the ring of fig. 1; the second may be represented by a sphere with a toroidal hole inside. Each of these is doubly connected, and has the property that a closed circuit can be described in the medium, and this circuit cannot shrink to a point while always remaining inside the medium. In either case a single cut removes this property. For the torus, a suitable cut is that made by an axial plane, such as AA′ in fig. 1 and the resulting body is topologically a sphere. For the toroidal hole, the cut is made outwards from the hole to the outer surface (fig. 12) and the result is again a topological sphere.

Fig. 12



A body with a toroidal hole of small bore is made singly connected by a cut. M_0 and M_1 are neighbouring points on either side of the cut, N_0 and N_1 are a second pair of points separated by the same cut.

In either case the body may be dislocated by making a cut (which need not be plane), displacing the two cut surfaces and possibly adding or removing thin layers of the medium, and finally joining the surfaces again in their disturbed positions.

Weingarten (1901) has shown that if the strains in the deformed body are everywhere finite and twice differentiable, the disturbance of the cut faces cannot be arbitrary, but their relative displacement must be one which is a possible displacement for a rigid body. The proof follows

(Love, p. 223) from the discussion of § 2.3. We consider the body of fig. 12. The surface PQR of fig. 11 is now replaced by the toroidal hole, and the circuit is one which cannot shrink to a point without leaving the medium. We take M_0 to be a point lying close to the cut surface, and M_1 the corresponding point on the other side of the cut. Similarly N_0 is a different point on the same side of the cut as M_0, and N_1 is the neighbouring point on the other side of the cut. We join M_0 to N_0 by a path lying in the medium and close to the cut, and M_1 to N_1 by a neighbouring path on the other side of the cut. From eqn. (21)

$$u\ (\mathrm{N}_0)-u_i(\mathrm{M}_0)=-\tfrac{1}{2}\{x_j(\mathrm{N}_0)-x_j(\mathrm{M}_0)\}\left\{\frac{\partial u_j}{\partial x_i}(\mathrm{M}_0)-\frac{\partial u_i}{\partial x_j}(\mathrm{M}_0)\right\}$$
$$+\int_{\mathrm{M}_0}^{\mathrm{N}_0}\xi_{ik}\,dx_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24)$$

and

$$u_i(\mathrm{N}_1)-u_i(\mathrm{M}_1)=-\tfrac{1}{2}\{x_j(\mathrm{N}_1)-x_j(\mathrm{M}_1)\}\left\{\frac{\partial u_j}{\partial x_i}(\mathrm{M}_1)-\frac{\partial u_i}{\partial x_j}(\mathrm{M}_1)\right\}$$
$$+\int_{\mathrm{M}_1}^{\mathrm{N}_1}\xi_{ik}\,dx_k\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

Now in (24) and (25)

$$x_j(\mathrm{M}_1)=x_j(\mathrm{M}_0)$$
$$x_j(\mathrm{N}_1)=x_j(\mathrm{N}_0),$$

the paths of the two integrals coincide, and ξ_{ik} has the same values at corresponding points of the two paths because the strain components are continuous. Subtracting (24) from (25), we obtain

$$u_i(\mathrm{N}_1)-u_i(\mathrm{N}_0)=[u_i(\mathrm{M}_1)-u_i(\mathrm{M}_0)]-\tfrac{1}{2}\{x_j(\mathrm{N})-x_j(\mathrm{M})\}$$
$$\times\left[\left\{\frac{\partial u_j}{\partial x_i}(\mathrm{M}_1)-\frac{\partial u_i}{\partial x_j}(\mathrm{M}_1)\right\}-\left\{\frac{\partial u_j}{\partial x_i}(\mathrm{M}_0)-\frac{\partial u_i}{\partial x_j}(\mathrm{M}_0)\right\}\right]. \quad . \quad . \qquad (26)$$

The expressions in square brackets in (26) are independent of the coordinates of N, and we see that the relative displacement $\delta u_i(\mathrm{N})$ of neighbouring points on either side of the cut at N is given by an expression of the form

$$\delta u_i(\mathrm{N})=b_i+d_{ij}x_j(\mathrm{N}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

in which $d_{ji}=-d_{ij}$. This represents an arbitrary rigid-body displacement.

The theorem, then, shows that in a multiply connected body we may make a cut, not necessarily plane, which renders the body singly connected. If the body is free from strain, the cut surfaces will remain in equilibrium in their original positions. Keeping one surface fixed, we may give the other any small rigid displacement, remove any surplus material where the surfaces overlap and fill in any gaps, seal the cut again, and release the applied stresses. Then the strains in the resulting body are continuous. If the second surface is deformed in any way which does not reduce to a rigid displacement, the strains are not finite and twice differentiable.

This is a very remarkable result, and appears to contradict intuitive ideas that a lenticular hole could be excavated in an elastic solid and the faces then welded together without difficulty. The theorem holds only on the assumption that the strain is continuous and twice differentiable. As an example of an apparent exception to the theorems of §§ 2.3 and 3.1, we may consider again the ring of fig. 1. If the ring is thin, so that AA′ is much less than OA, the radial displacement is clearly given to a good approximation by $u_r = r\phi/2\pi$, and it is not immediately obvious why the dislocation shown can only be effected in a multiply connected ring and not in a solid cylinder. The exact solution (Love, p. 227), shows that u_r contains a term proportional to $r \ln r$. The strain is therefore infinite at the origin.

The result has been proved for the doubly connected body and the type of cut shown in fig. 12, but it is equally valid for the torus of fig. 1, or for the body of fig. 12 if a cut is made *inwards* from the toroidal hole, so that the cut body has the connectivity of a hollow sphere. If the toroidal hole is of very small bore, the cut is plane, and the relative displacement of the cut surfaces is one atomic distance in a direction lying in their plane, the resulting strain is that produced by Orowan's loop dislocation (fig. 4).

In the present mode of description, the dislocation of fig. 1 was made by cutting along AA′, keeping the left face of the cut fixed, and rotating the right face rigidly about O until the line BB′ fell on AA′. The wedge AA′ B′B was then removed. It was clear that the ring could be brought into the same state of strain by making any radial cut such as CC′. Volterra (1907) has proved that any dislocation of Weingarten's type is equivalent to the dislocation produced by applying the same translation and rotation to the surfaces of any cut which can be continuously deformed into the original cut.

3.2. THE SIX SIMPLE TYPES OF DISLOCATION

According to eqn. (27), the most general dislocation of a doubly connected body is described by three constants b_i specifying the relative translation of the two cut surfaces, and three constants d_{ij} specifying their relative rotation. Each of these six may be regarded as an independent dislocation, and the corresponding constant is the strength of this elementary dislocation. For a closed ring (not necessarily a body of rotation) encircling the z axis, displacements which will maintain equilibrium under no body forces, though surface forces are required, are given by

$$\left.\begin{aligned} 2\pi u_x &= (b_x - d_{zx}z + d_{xy}y)\tan^{-1}(y/x) + \tfrac{1}{2}(-b_y - d_{yz}z - \alpha d_{xy}x)\ln(x^2+y^2) \\ 2\pi u_y &= (b_y - d_{xy}x + d_{yz}z)\tan^{-1}(y/x) + \tfrac{1}{2}(b_x - d_{zx}z - \alpha d_{xy}y)\ln(x^2+y^2) \\ 2\pi u_z &= (b_z - d_{yz}y + d_{zx}x)\tan^{-1}(y/x) + \tfrac{1}{2}(d_{yz}x + d_{zx}y)\ln(x^2+y^2), \end{aligned}\right\} \quad (28)$$

where $\alpha = \mu/(\lambda + 2\mu)$.

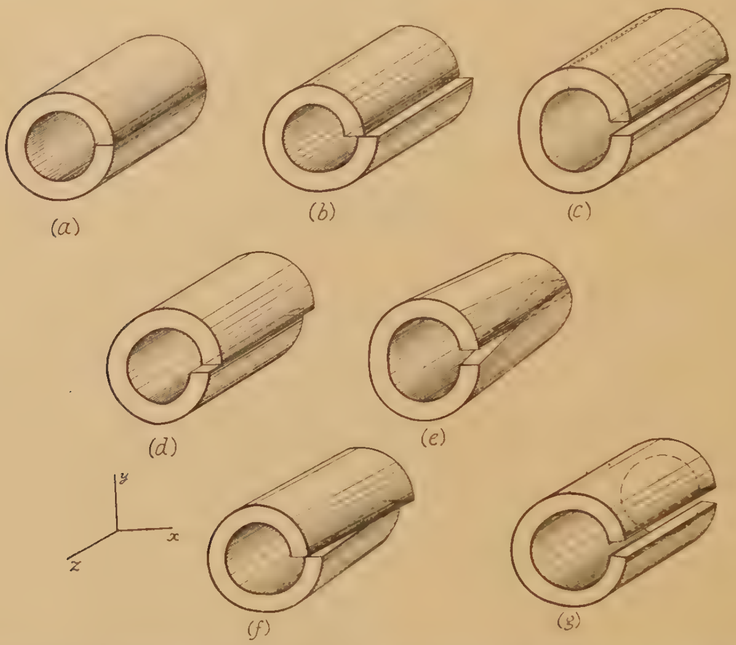
Here λ is Lamé's elastic constant, related to the rigidity μ and Poisson's ratio σ by the expression

$$\lambda = 2\sigma\mu/(1 - 2\sigma). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Volterra has discussed in detail the disturbance produced by each of these six types of dislocation in a hollow circular cylinder. If we take the origin of coordinates at the centre of the cylinder, the axis of the cylinder as z axis, and the positive x axis lying in the cut, the six orders of dislocation are those sketched in fig. 13.

The first order, b_x, is the edge dislocation. The difference between the displacements given by (28) and those given by (9) is caused by the neglect of surface forces in the derivation of (28). The second order, b_y, is identical with an edge dislocation along the y axis, as may be seen either from the form of eqns. (28) or by considering the cylinder cut and displaced along the dotted line of fig. 13 (c) instead of the full line of fig. 13 (a). This is an example of the theorem that two dislocations are equivalent if they represent the same rigid-body displacement of the faces of cuts which can be continuously deformed into each other. The third order, b_z, is the screw dislocation of § 2.2. The components of displacement in eqns. (28) and (11) agree. In the last three orders, the residue of the displacement integrated round a circular circuit is not independent of the radius of the circle as in the first three orders, but proportional to the radius, and the strain does not tend to zero at infinity.

Fig. 13



The six simple types of dislocation. (a) Original cut cylinder. (b) Dislocation of type b_x. (c) Type $-b_y$. (d) Type b_z. (e) Type $-d_{yz}$. (f) Type $-d_{zx}$. (g) Type d_{xy}.

We are therefore not concerned with these dislocations in the theory of plastic flow. Here again the dislocations of the fourth and fifth orders are equivalent. The dislocation of the sixth order is that shown in fig. 1.

3.3. MORE GENERAL DISLOCATIONS

Mann (1949) and Bogdanoff (1950) have recently analysed dislocations of a more general type than those discussed by Weingarten and Volterra. They allow the tangential components of stress to be discontinuous across a plane cut, while the tractions across the cut are still continuous. The resulting stress system is of the type occurring in Eshelby's analysis of the core of an edge dislocation, or in a slip band in which dislocations are still present in the slip plane. It may be regarded as arising from the superposition of a continuous distribution of simple dislocations. Mann considered stress functions of the form ln r or θ multiplied by

$$r^{m+1}\,e^{(m-1)i\theta}. \qquad \tag{30}$$

Bogdanoff showed that the general solution requires also three stress functions which were disclosed to him by R. D. Mindlin, namely

$$\theta^2-(\ln r)^2 \qquad \tag{31}$$

and the same factor multiplied by $r\cos\theta$ and $r\sin\theta$.

Eshelby (1951) has considered the case in which the medium contains a continuous distribution of dislocations. The elastic conditions of compatibility may be regarded (Love, p. 50) as expressing the vanishing of a tensor

$$S_{rs}=e_{rs,ii}+e_{ii,rs}-e_{ri,si}-e_{si,ri}-(e_{ii,jj}-e_{ij,ij})\delta_{rs}. \qquad \tag{32}$$

If we consider a non-Euclidean space defined by the metric

$$g_{ij}=\delta_{ij}+2e_{ij}, \qquad \tag{33}$$

its Riemann tensor is given to first order in e_{ij} by

$$R_{prst}=\epsilon_{ipr}\epsilon_{jst}S_{ij}. \qquad \tag{34}$$

The vanishing of S_{ij} is thus the condition that the space (33) should be flat. The diagonal terms of S_{ij} represent the distribution of infinitesimal dislocations of the type of fig. 4, the non-diagonal terms the distribution of dislocations of the type of fig. 15.

3.4. BURGERS'S THEORY OF LINE DISLOCATIONS

Burgers (1939 a) has developed Volterra's theory of dislocations for the case which is of importance for the theory of slip, that is, dislocations of the first three orders in solids from which filaments of matter have been removed in such a way that the radius of curvature of the filament is everywhere large compared with its diameter. He considered a closed filament, corresponding to the toroidal hole of fig. 12, in an isotropic medium. A dislocation is made by a relative displacement $\mathbf{b}$ of the two surfaces of a cap whose edge is formed by this loop. The loop is known as the *line* of the dislocation: it is convenient to have a name for the vector **b**, and we shall call it the *Burgers vector* of the dislocation. The strain at any point depends only on **b** and on the position of the line of the dislocation, and is independent of the form of the cap. Burgers illustrates this fact by expressing the displacement **u** at a point **x** by an integral round the line of the dislocation, together with a multiple valued term proportional to the hydrodynamic potential at **x** of a vortex of unit strength along the line of the dislocation. The complete solution is given by

$$\mathbf{u}=\mathbf{u}^{*}+\mathbf{u}^{**}+\operatorname{grad}\psi. \qquad \qquad (35)$$

Here

$$\mathbf{u}^{*}=\Omega\mathbf{b}/4\pi,$$

where Ω is the solid angle (a multiple valued quantity with residue 4π) subtended by the dislocation line at **x**,

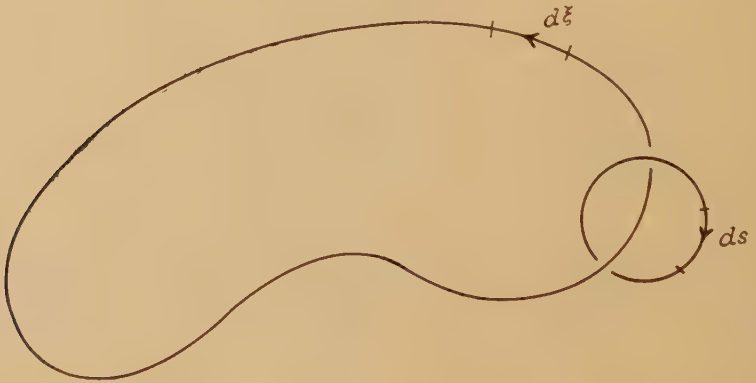
$$\mathbf{u}^{**}=\frac{\mathbf{b}}{4\pi}\wedge\int\frac{d\boldsymbol{\xi}}{r},$$

where $d\boldsymbol{\xi}$ is an element of the line of the dislocation, $r=|\mathbf{x}-\boldsymbol{\xi}|$, and the integral is taken once round the dislocation line in an arbitrary sense, and

$$\psi=\frac{\lambda+\mu}{4\pi(\lambda+2\mu)}\int\mathbf{b}\wedge(\mathbf{x}-\boldsymbol{\xi})\frac{d\boldsymbol{\xi}}{r}.$$

In this solution, the force and acting couple on unit length of the filament are of the same order as its radius, and so are negligible.

Fig. 14



Sign convention for the Burgers vector. The positive direction on the line of the dislocation $d\xi$ is chosen arbitrarily. The Burgers vector is minus the residue of displacement round a right handed circuit $\int ds$.

The signs of Taylor's two dimensional dislocations can be established by one of the usual conventions of contour integrals. Orowan's loop dislocation appears in section as a pair of edge dislocations of opposite sign, and a convention is required to define the sign of its Burgers vector. We adopt the convention that the positive direction $d\xi$ of the circuit of the dislocation line (fig. 14) is chosen arbitrarily.

Then the Burgers vector is given by $\mathbf{b}=-\int(\partial\mathbf{u}/\partial s)\,ds$, where the integral is taken once round a curve which encircles the dislocation line with a right-handed screw. Bilby (1951) has discussed this rule.

Burgers (1939 b) has extended his results to the case of a dislocation loop in a cubic crystal. The general formulae are complicated, but in the case of screw dislocations with their lines along a cube axis the displacements are the same as those in an isotropic medium, while there is also a considerable simplification for edge dislocations along a cube axis.

Eshelby (1949 a) has obtained the solution for an infinite straight edge dislocation in a crystal of any symmetry. His general formulae are explicit, but necessarily rather clumsy. They simplify in a number of important cases for which, owing either to the symmetry of the crystal or to the special direction of the dislocation line, the ellipses of stress and strain in the plane of the strain have their axes parallel. He has applied his result to estimate the energies of edge dislocations lying along various lines (cf. § 2.1 of Part III), and their widths and mobilities (cf. § 5.3 of Part III).

Nabarro (1951 b) has evaluated Burgers's formulae for an isotropic medium for the special case of an infinitesimal loop of the type of fig. 4. Since (§ 4.1) any glide motion may be considered to be produced by the creation of such infinitesimal loops, the results may be used to synthesize the stress field of any dislocation which could have been caused by glide. The method does not seem to be of much value for stationary dislocations, but it is the only general method available for moving dislocations.

3.5. THE ENERGIES OF ELASTIC DISLOCATIONS

We consider an isolated elastic body containing a number of holes, each of the type shown in fig. 12. The body is rendered singly connected by a set of surfaces Σ_s, each bounded by one hole, and dislocations b_{is}, d_{ijs} (eqn. (27)) are made on each surface. Suppose that when the dislocations are made, the force acting on an element $d\Sigma_s$ of that surface of the cut Σ_s which suffers the displacement

$$u_{is}=b_{is}+d_{ijs}x_j$$

has components $X_i\,d\Sigma_s$. Then the energy E of the system is easily seen to be

$$E=\tfrac{1}{2}\iint(b_{is}+d_{ijs}x_j)X_i\,d\Sigma_s, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where summations over all the repeated suffixes are implied. This formula may be employed to calculate the energy of interaction of two dislocations. Since b_{is} and d_{ijs} are constants for each surface, they may be taken outside the integrals. Each integral then represents a component of the resultant force or couple transmitted across a surface Σ_s. Alternatively, the total energy may be calculated by integration if the components of strain and thus the energy density are known at all points of the body.

It has already been observed that the energy for unit length of an edge or a screw dislocation in an infinite medium diverges logarithmically at infinity. In practical applications this difficulty is usually overcome by cutting off the integration at a distance which may be determined by the average separation of dislocations or some similar consideration. For a dislocation loop the difficulty does not arise, as the integrals in (36) are clearly finite. Alternatively, we observe that, at distances large compared with the dimensions of the loop, the displacement corresponds to that caused by a 'double force' (Love, p. 187), and so is of order r^{-2}. The strain is of order r^{-3}, and the energy is finite.

Suppose now that the multiply connected body is subjected to external forces, which are held constant while the dislocations are formed. If *small* dislocations are formed, the forces acting on $d\Sigma_s$ are the sum of those $X_i\,d\Sigma_s$ produced by the dislocations and the constant $X_i'\,d\Sigma_s$ produced by the external forces. These X_i' are derived from the *actual* stresses over Σ_s, so that even if the stress in the body is uniform far from the dislocations, it will be non-uniform over Σ_s because of the stress concentration produced by the hole surrounding Σ_s. Work is also done by the external forces when the dislocations are made, because the outer boundary of the body deforms. Let this work be W_e. Both W_e and the work done by the forces acting across Σ_s when the dislocation is made are linear in the strength of the dislocations, while the self energy E of the system of dislocations is (eqn. (36)) quadratic in their strength, and may be neglected. The work done in making the small dislocations is thus

$$W_e + \iint (b_{is}' + d_{ijs}' x_j) X_i \, d\Sigma_s, \qquad \ldots\ldots \quad (37)$$

where b_{is}' and d_{ijs}' are the strengths of the dislocations, and X_i are the forces acting over Σ_s due to the external forces.

If ϕ is the strain energy density at any point, ϕ is a function of the strains e_{ij} produced by the external forces, and the work done in making a small dislocation may be written

$$\iiint \frac{\partial \phi}{\partial e_{ij}} e_{ij}' \, dV. \qquad \ldots\ldots\ldots \quad (38)$$

Here e_{ij}' are the strains produced by the dislocations, and the integral is extended over the volume of the body.

Now consider the same body under no external forces, but with dislocations already formed. If arbitrary small strains e_{ij} which do not form dislocations are imposed on the body, no work is done by external forces or by forces over the surfaces Σ_s, and so by the principle of virtual work there is no change in the stored energy. The strain is now e_{ij}', and we find

$$\iiint \frac{\partial \phi'}{\partial e_{ij}'} e_{ij} \, dV = 0. \qquad \ldots\ldots\ldots \quad (39)$$

In particular, we may take the e_{ij} in (39) proportional to the strains produced by the external forces in (38). The function ϕ is homogeneous and quadratic in the strains, and the integral in (39) may be transformed by the relation

$$\frac{\partial \phi'}{\partial e_{ij}'} e_{ij} = \tfrac{1}{2} e_{ij} e_{rs}' \frac{\partial^2 \phi}{\partial e_{ij}\, \partial e_{rs}} = \frac{\partial \phi}{\partial e_{rs}} e_{rs}'.$$

It follows that eqn. (39) implies the vanishing of expression (38). Expression (37) must also vanish, and the work done by the external forces when the dislocation is made is

$$W_e = -\iiint (b_{is}' + d_{ijs}' x_j) X_i \, d\Sigma_s.$$

This important theorem is due to Colonnetti (1915), and is of value in classifying the motions of dislocations (§ 4.1) and in the study of the dynamics of dislocations. For the dislocations of the first three orders which are important in plasticity we may write

$$W_e = \iint b_{is}' p_{ij} \, d\Sigma_{js}, \qquad \text{. . . . . (40)}$$

where p_{ij} is the stress produced by the external forces, and $d\Sigma_{js}$ is a component of $d\Sigma_s$, regarded as a vector. The sign of the vector is positive if it is derived from the circuit $\int d\xi$ of fig. 14 by a right handed screw rule.

Following a suggestion of Leibfried (1949), we may interpret Colonnetti's theorem as follows. The vanishing of (37) means that the work done in forming dislocations is independent of the external stresses. If we express the total strain energy as a quadratic function of the strengths of the dislocations and of the applied stresses, there are no cross terms between dislocations and applied stresses. This means that as long as the dislocations remain constant and Hooke's Law remains valid, we cannot detect the presence of dislocations by observing the response of the body to external forces.

§4. The Motion of Dislocations

4.1. POSSIBLE MOTIONS OF A DISLOCATION

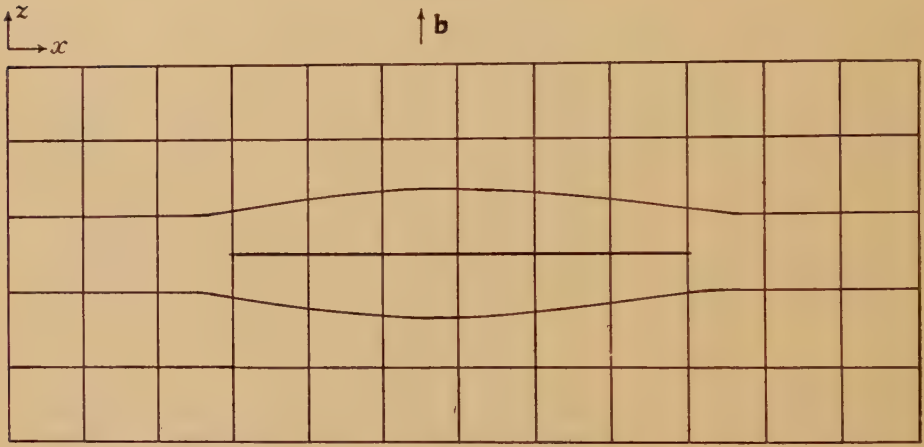
The possible motions of a dislocation may be classified (Nabarro 1951 a) by means of formula (40). Suppose a single dislocation is formed in a solid under a uniform pressure P, so that the stress is given by $p_{ij} = -\delta_{ij} P$. Here $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise. This is the stress at all points of Σ, except those within a distance of order b of its boundary. The energy of interaction in this boundary region is of order $b^2 PL$, where L is the length of the dislocation, and may be neglected if the area of the dislocation is large compared with b^2. For a large dislocation loop, the work done by the external pressure P when the loop is formed is therefore

$$\begin{aligned} W_e &= P \iint b_i \, d\Sigma_i \\ &= -P \iint \mathbf{b} \, d\mathbf{\Sigma}, \qquad \text{. . . . . (41)} \end{aligned}$$

and $\iint \mathbf{b} \, d\mathbf{\Sigma}$ is the increase in volume of the body when the dislocation is formed. Since (Love, p. 175) in the absence of surface or body forces the mean value of the dilatation in the dislocated body is zero, $\iint \mathbf{b} \, d\mathbf{\Sigma}$ must be the volume of matter inserted into the body when the dislocation was formed. For the loop dislocation of fig. 4, **b** and $d\boldsymbol{\Sigma}$ are orthogonal, and $\iint \mathbf{b}\, d\boldsymbol{\Sigma}=0$. If an extra layer of atoms of thickness b and having the same boundary had been inserted into the lattice (fig. 15), the resulting dislocation would be characterized by $\iint \mathbf{b}\, d\boldsymbol{\Sigma}=b\times$(area of loop), while for an equal number of atoms removed $\iint \mathbf{b}\, d\boldsymbol{\Sigma}$ would be negative.

In normal glide motions the dislocation line always moves on a cylindrical surface with the vector **b** as axis, and both **b** and the position and shape of the projection of Σ on a plane perpendicular to **b** are conserved independently. However (Frank 1951 a), this projection may add segments of zero area to its periphery. For example, the dislocation of fig. 15 is everywhere of edge type, and each part of it can only move in the z direction. The parts which have moved are now joined to the parts which have not moved by segments of screw type, and these can glide freely over any cylindrical surface with **b** as axis. The dislocation is shown in perspective as ABCDA in fig. 16 (a), and its projection on the xy plane in fig. 16 (b). Its movement to the form ABEFDA does not alter its projection. In fig. 16 (c) the screw segment BE has sent out an open loop BGHJE, and the xy projection is now that shown in fig. 16 (d). The loop GHJ, which projects into a single line in the plane perpendicular to **b**, is similar to a segment of the dislocation of fig. 4.

Fig. 15

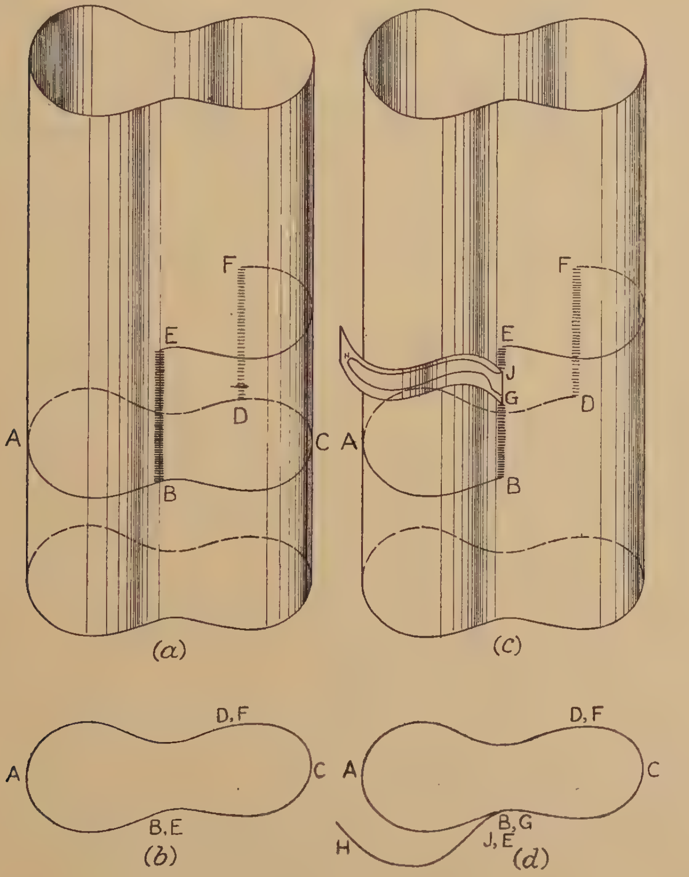


Section of a loop of dislocation with Burgers vector **b** normal to the plane of the loop (xy plane).

It is possible for the value of $\iint \mathbf{b}\, d\boldsymbol{\Sigma}$ to be conserved without preserving the position and shape of the projection of the dislocation line perpendicular to **b**. For example, the added sheet of atoms in figs. 15 and 16 can move bodily in the xy plane, first gliding over the plane of atoms below and then allowing the plane of atoms above to glide back.

The process does not use the existing dislocation **b** with its vector in the z direction, but involves the formation of new loops of dislocation with Burgers vectors in the xy plane, for which $\iint \mathbf{b}\, d\mathbf{\Sigma}=0$. If only a part of the area ABCDA glides in this manner, the shape as well as the position of Σ alters. It is also geometrically possible for part of Σ to glide over another part of Σ, producing a region with Burgers vector 2**b**.

Fig. 16



Possible glide motions of a dislocation. (*a*) Loop of dislocation ABCDA, lying on a cylinder with axis parallel to the Burgers vector **b**. The segment BCD can glide to EF, leaving screw segments BE, DF which join it to DAB. (*b*) Projection of (*a*) normal to the Burgers vector. (*c*) A loop GHJ can spread from the screw segment BE, provided that it lies on a cylindrical surface with **b** as generator. (*d*) Projection of (*c*) perpendicular to **b**.

These movements are not of direct importance in plastic deformation, because they do not do work in a uniformly stressed body. In normal glide motions the projections of Σ on planes passing through **b** alter, and so, from eqn. (40), work is done by shear stresses ($i \neq j$), but in these motions all the projections of Σ are unaltered. A rigid motion of the curve ABCDA up its cylinder is also of this ineffective kind, although it does not require the formation of new loops of dislocation.

The third possible type of motion is a non-conservative motion in which the value of $\iint \mathbf{b}\, d\boldsymbol{\Sigma}$ alters. This involves the movement of segments of edge type in a direction perpendicular to **b**, and occurs by the diffusion of vacant lattice sites or of interstitial ions to the line of the dislocations (Nabarro 1948 b). When one vacant site has reached the edge of the dislocation in each lattice plane, the dislocation line has moved by one lattice spacing perpendicular to b, as may be seen from figs. 6 and 15. Such a process is slow, and its rate depends on temperature in the same way as the coefficient of self diffusion. The opening of a crack in a body may also be regarded as a non-conservative motion of a dislocation line. Non-conservative processes, which introduce an important element of irreversibility into plastic deformation, have been discussed in detail by Seitz (1950 b, 1952) and Mott (1952 a).

4.2. THE FRANK–READ SOURCE

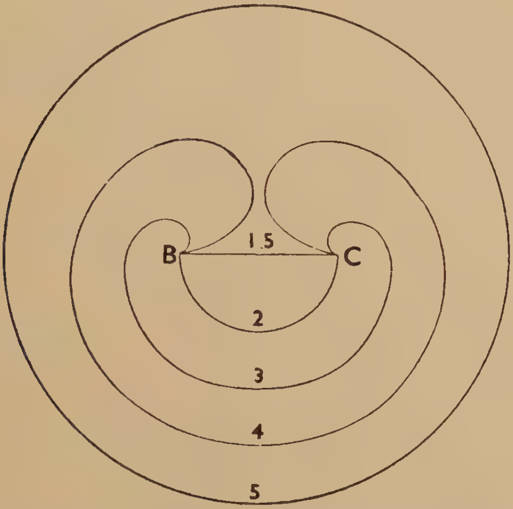
Frank and Read (1950) have introduced a type of motion of a dislocation which has profoundly influenced the development of the theory of slip. Previously attention had been concentrated on dislocations lying entirely in one plane. The growth of such a loop in the glide plane causes slip, but at the same time the dislocation passes out of the region under consideration and ultimately passes right out of the specimen at its surface. If two portions of the loop happen to meet, they annihilate one another. There is thus no explanation for the large amount of shear which is experimentally observed to occur on a single glide plane. Frank (1948) showed that fast moving dislocations might multiply when they struck the free surface or other fast dislocations, but it seems clear (Leibfried 1950, Nabarro 1951 c) that such processes can occur only in the upper part of the range of stresses which in practice are applied to metals.

Consider now the loop of dislocation ABCD of fig. 16, which does not lie in one glide plane, and suppose that owing to the details of the crystal structure, e.g. the possibility of dissociation in certain planes (§ 5.2 of Part III) the loop forms a rectangle with corners ABCD. Suppose moreover that the segments AB and CD are anchored, while glide can occur in the planes containing the slip vector **b** and one of the lines BC, DA. Then each of the segments BC and DA can move in its glide plane without interference from the other segments. Figure 17 illustrates what happens in the glide plane of BC when a stress is applied. The original line 1 bulges to a loop 2, then to 3. At 4, two elements of the loop are about to meet. They annihilate, and the next configuration is a closed loop 5 together with a short segment of dislocation joining B and C. The loop now expands freely, and is ultimately lost, while the short segment again starts the sequence 2, 3, 4 . . . The original segment 1, which can give rise to an unlimited succession of closed loops of dislocation, is called a Frank–Read source.

4.3. THE FORCE ON A DISLOCATION : LAW OF CONSTANT RESOLVED SHEAR STRESS

Taylor's expressions for the stresses round an edge dislocation in an elastic continuum require a resultant force to act on any small closed curve surrounding the dislocation, whereas those given by Burgers do not, and for this reason we have rejected Taylor's solution. Suppose now that a dislocation is in equilibrium in an elastic medium, and a shear stress is applied which is uniform in the parts of the medium far from the dislocation. There is, in the elastic solution, a concentration of stress near the filamentary hole surrounding the dislocation, but the resulting stress distribution does not require forces to be applied across the surface of this hole, and in this sense the dislocation is still in equilibrium, with no force acting on it. In a crystal lattice, the line of the dislocation is no longer fixed, and the energy of the system may alter if the dislocation moves. It is now convenient to use the expression ‘ force acting on the dislocation ’ in another sense. If the displacement of a segment of the dislocation through a small distance **d** with components d_i

Fig. 17



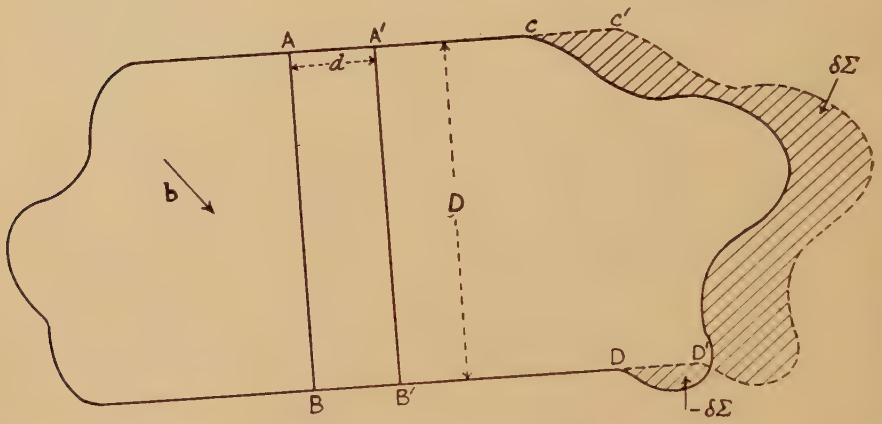
The Frank–Read source. A segment BC of dislocation lies in the glide plane, which is the plane of the paper. At B and C it turns out of the plane, and is held fixed. Under an applied stress the mobile segment BC moves from 1 to 2, then to 3, 4 and finally to 5. It now consists of a large loop together with the original segment BC, which can again repeat the sequence 1, 2, 3. . . .

causes the energy of the whole system, including the source of the applied stresses, to decrease by an amount U, we say that the force acting on the segment has components F_i given by

$$F_i = \frac{\partial U}{\partial d_i}. \qquad \ldots\ldots\ldots (42)$$

There are three contributions to the quantity dU, which correspond to the changes in the self energy of the dislocation, in its energy of interaction with other dislocations, and in its energy of interaction with the applied stresses. The first two are given by $-dE$, where E is given by eqn. (36), and the third is dW_e (eqn. 40). In interpreting these formulae, X_{is} and p_{ij} must be taken as the forces and stresses actually present on Σ_s. Because Σ_s is bounded in the elastic model by a cavity, and in the crystal lattice by a region where Hooke's law does not hold, the values to be taken in calculating the interaction energies differ from those which would hold if there were no cavity and no dislocation surrounding Σ_s, and these values alter when the dislocation moves. As in the derivation of eqn. (41), we consider only large loops of dislocation, and restrict attention to displacements in which segments having small radii of curvature move rigidly. As an example consider the dislocation of fig. 18, partly bounded by two parallel straight lines. If the part CD of the boundary moves by a glide motion parallel to these lines into a new position C′D′, the area of the dislocation is increased by $\delta\Sigma$. Since this is a glide motion conserving $\iint \mathbf{b}\, d\mathbf{\Sigma}$, the Burgers vector $\mathbf{b}$ must lie in the plane of the figure. The shift has added two new segments AA′, BB′ to the boundary. Their self energy is of order $b^2\mu d$, where d is the distance AA′, assumed much less than the perpendicular distance D between the lines. Owing to the increased length of the boundaries, which cause stress concentrations, the energy of the applied stress P

Fig. 18



When the segment of dislocation CD moves to C′D′, the total area swept out is equal to AA′B′B.

changes by an amount of order b^2P^2d/μ, and the interaction energy of P and the stress field round the dislocation changes by b^2Pd. Since P is always less than μ, the first of these three terms is the largest. The dominant term in dW_e is of order $bPdD$, and the change in self energy is negligible in comparison with this provided $D/b \gg \mu/P$. For a soft single crystal μ/P may be of the order 5.10^4, and the self energy of the dislocation is negligible only for loops with $D/b > 10^5$, that is, for loops 10^5 atoms across. In harder materials this figure is reduced to a few hundreds of atoms.

Along CD and C′D′ there is no change of self energy. The interaction energy of P and the dislocation stress is of order b^2PD, but the contributions from CD and C′D′ are nearly equal, their difference being of order b^2Pd and so negligible. Provided that the stresses would in the absence of this dislocation have been constant on the surface Σ, we may consider the added area to be the equal rectangular area AA′B′B and substitute in eqn. (40) the unperturbed values of p_{ij} on this rectangle. In the general case of a change in the boundary of a small loop we shall again take the unperturbed values of p_{ij}, but we take the actual new area $\delta\Sigma$ near the boundary.

If an element $d\boldsymbol{\xi}$ of the dislocation (fig. 14) moves a distance $d\mathbf{x}$ the area of the dislocation increases by

$$\delta\boldsymbol{\Sigma}=d\mathbf{x}\wedge d\boldsymbol{\xi}. \qquad (43)$$

We write (40) in the form

$$\begin{aligned}\delta W&=\mathbf{b}\boldsymbol{\mathfrak{P}}\delta\boldsymbol{\Sigma}\\&=\mathbf{b}\boldsymbol{\mathfrak{P}}\,.\,(d\mathbf{x}\wedge d\boldsymbol{\xi})\\&=[(\mathbf{b}\boldsymbol{\mathfrak{P}})\wedge d\boldsymbol{\xi}]\,.\,d\mathbf{x}. \qquad (44)\end{aligned}$$

The force $d\mathbf{F}$ on the element $d\boldsymbol{\xi}$ is thus given by

$$d\mathbf{F}=(\mathbf{b}\boldsymbol{\mathfrak{P}})\wedge d\boldsymbol{\xi}. \qquad (45)$$

This result was first obtained by Peach and Koehler (1950).

In the definition of elastic dislocations from which this formula was derived, the creation or removal of matter was permitted, and (45) therefore gives the total force on the element, including the component which would lead to a non-conservative motion. To obtain the force $\mathbf{f}$ leading to glide, we subtract the component of $\mathbf{F}$ normal to the glide plane. The normal to the glide plane is the unit vector $\mathbf{n}$ defined by

$$\mathbf{n}=(\mathbf{b}\wedge d\boldsymbol{\xi})/|\mathbf{b}\wedge d\boldsymbol{\xi}|. \qquad (46)$$

The component $d\mathbf{f}$ of the force $d\mathbf{F}$ which is effective in producing slip is the component lying in the glide plane, given by

$$d\mathbf{f}=d\mathbf{F}-\mathbf{n}(\mathbf{n}d\mathbf{F})=-\mathbf{n}\wedge(\mathbf{n}\wedge d\mathbf{F}). \qquad (47)$$

Now

$$\begin{aligned}\mathbf{n}\wedge d\mathbf{F}&=\mathbf{n}\wedge[(\mathbf{b}\boldsymbol{\mathfrak{P}})\wedge d\boldsymbol{\xi}]\\&=(\mathbf{b}\boldsymbol{\mathfrak{P}})(\mathbf{n}d\boldsymbol{\xi})-d\boldsymbol{\xi}[\mathbf{n}(\mathbf{b}\boldsymbol{\mathfrak{P}})]\\&=-d\boldsymbol{\xi}(\mathbf{b}\boldsymbol{\mathfrak{P}}\mathbf{n}),\end{aligned}$$

and so

$$d\mathbf{f}=(\mathbf{n}\wedge d\boldsymbol{\xi})(\mathbf{b}\boldsymbol{\mathfrak{P}}\mathbf{n}). \qquad (48)$$

Since $\mathfrak{P}\mathbf{n}$ is the force acting across unit area of the glide plane, the force per unit length tending to produce glide is b times the component in the glide direction of the force acting across unit area of the glide plane, and acts in the glide plane normal to the line of the dislocation. If a loop of dislocation lies entirely in the glide plane, its conservative motions are confined to this plane (except for those elements which are of pure screw type), and its whole length is subjected to the same normal force tending to increase the area of the loop. This is in accordance with the experimental result (Schmid 1924) that slip begins when the ‘ resolved shear stress ’ $\mathbf{b}\mathfrak{P}\mathbf{n}/b$ reaches a value which is characteristic of the crystal, but independent of the other components of the applied stress.

The expression (48) differs from that given by Koehler (1941) by a numerical factor. Koehler calculated the work done by a uniform shear stress when the dislocation moves by integrating the strain energy within a large region, and then slightly displacing the boundary of the region over which the integral is taken. He did not allow for the difference between the displacements produced by the dislocation at a free boundary and the displacements at the corresponding points of an infinite medium (cf. § 4.4). The correct result was stated for a special case by Mott and Nabarro (1948) and proved by Leibfried (1949) and Read and Shockley (1950). When one considers the force between two dislocations or between a dislocation and an inhomogeneity of a different kind, the chances of error and confusion are enhanced. The position has been summarized by Eshelby (1951) and Peach (1951).

4.4. EXPLICIT CALCULATION OF THE FORCE ON A DISLOCATION

Eshelby (unpublished) has given detailed calculations of the forces on an edge dislocation lying along the axis of a cylinder of infinite length and finite radius, and on a screw dislocation lying along any line parallel to the axis of such a cylinder. Nabarro (1951 b) has considered the work done in forming an infinitesimal loop of dislocation at the centre of a homogeneously stressed sphere. The results are all in agreement with (48).

4.41. *Force on an Edge Dislocation*

The displacements produced by an edge dislocation lying in an infinite block along the z axis are given by (8), and the corresponding stress function χ^0 is given by (1) and (4). In polar coordinates the stresses are given by

$$\chi^0 = -Dr \ln r \sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

$$\left.\begin{aligned} p_{rr}{}^0 &= -Dr^{-1}\sin\theta \\ p_{r\theta}{}^0 &= Dr^{-1}\cos\theta \\ p_{\theta\theta}{}^0 &= -Dr^{-1}\sin\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

and

where

$$D = b\mu/2\pi(1-\sigma). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

The tractions across a cylindrical surface of radius R with the dislocation as its axis do not vanish, and if this cylinder is cut out of the block and its surfaces are freed from stress the displacements alter. The new displacements are derived from the stress-function $\chi^0+\chi^1$, where

$$\chi^1=\tfrac{1}{2}DR^{-2}r^3 \sin\theta. \qquad (52)$$

Over the surface $r=R$ the tractions produced by χ^1 annul those produced by χ^0.

If now the dislocation is moved to the point $x=\delta\xi$, $y=0$, where $\delta\xi \ll R$, the stress function χ^0 changes to $\chi^0+\chi^2$, where

$$\begin{aligned}\chi^2&=-\delta\xi\,\partial\chi^0/\partial x\\ &=\tfrac{1}{2}D\delta\xi \sin 2\theta. \qquad (53)\end{aligned}$$

The corresponding stresses are

$$\left.\begin{aligned}p_{rr}{}^2&=-2D\delta\xi r^{-2}\sin 2\theta\\ p_{r\theta}{}^2&=D\delta\xi r^{-2}\cos 2\theta.\end{aligned}\right\} \qquad (54)$$

The stress function which annuls the tractions over $r=R$ is

$$\chi^3=-D\delta\xi R^{-2}r^2\sin 2\theta+D\delta\xi R^{-4}r^4\sin 2\theta, \qquad (55)$$

and the stress function for the cylinder with free surfaces and an edge dislocation slightly displaced from its axis is $\chi^0+\chi^1+\chi^2+\chi^3$. The displacement of the surface when the dislocation moves from (0, 0) to $(\delta\xi, 0)$ is the displacement due to $\chi^2+\chi^3$, which can be shown to be

$$u_r=b\delta\xi\sin 2\theta/\pi R, \quad u_\theta=0. \qquad (56)$$

A shear stress $p_{xy}=P$ produces a normal force $P\sin 2\theta$ on unit area of the cylinder, and the work done in the displacement (56) is

$$W_e=\frac{1}{\pi R}\int_{-\pi}^{\pi} bP\delta\xi \sin^2 2\theta\,.\,Rd\theta=bP\delta\xi.$$

The force on unit length of the dislocation is $\partial W_e/\partial\xi=bP$.

4.42. *Force on a Screw Dislocation*

The displacement and stress produced by a screw dislocation along the z axis are given by (15). For a parallel dislocation passing through the point $x=\xi$, $y=0$, the displacement is

$$u_z{}^0=\frac{b}{2\pi}\tan^{-1}\frac{y}{x-\xi}, \qquad (57)$$

while, on a circle of radius $R>\xi$ about the origin,

$$p^0_{rz}=-\frac{b\mu\xi}{2\pi}\,\frac{\sin\theta}{R^2-2R\xi\cos\theta+\xi^2}. \qquad (58)$$

The stress on the circle is annulled by a dislocation of the same strength and opposite sign at the inverse point $(R^2/\xi, 0)$, for which the displacement is

$$u_z{}^1=-\frac{b}{2\pi}\tan^{-1}\frac{y}{x-R^2/\xi}. \qquad (59)$$

Since on this circle

$$\frac{\partial u^0}{\partial\xi}=\frac{\partial u^1}{\partial\xi}=\frac{b}{2\pi}\,\frac{R\sin\theta}{R^2-2R\xi\cos\theta+\xi^2},$$

the total displacement of a free boundary when the dislocation moves from $(\xi, 0)$ to $(\xi+\delta\xi, 0)$ is

$$\delta u_z = \frac{b\delta\xi}{\pi} \frac{R \sin \theta}{R^2 - 2R\xi \cos \theta + \xi^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

If a uniform shear stress P, for which $p_{rz} = P \sin(\theta - \alpha)$, acts across the surface of the cylinder, the work done in this displacement is

$$W_e = \int_{-\pi}^{\pi} P \sin(\theta - \alpha) \delta u_z R d\theta = bP\delta\xi \cos \alpha.$$

The force is $bP \cos \alpha$, again corresponding to (48), whatever the initial value of ξ. If the dislocation moves across the cylinder at a constant rate, the external forces work at a constant rate. There is no sudden displacement as the dislocation enters or leaves the specimen.

4.43. *Force Expanding an Infinitesimal Loop of Dislocation*

The calculation is similar to that of §§ 4.41 and 4.42. The stress field of a loop in an infinite medium is obtained. To the corresponding displacements are added the displacements produced by forces which annul the tractions on the surface of a sphere of radius R large compared with the radius of the loop. If the loop is formed in the presence of a homogeneous external stress, the only component of this stress which does work is the component which gives a traction across the glide plane in the glide direction.

4.5. FORCE ON A DISLOCATION IN A VARYING STRESS FIELD

Suppose a dislocation is introduced into a body in which the stress field is not uniform. Then we wish to know if the force on each element of the dislocation is still given by (42), where $\mathfrak{P}$ is the applied stress in the neighbourhood of this element, or whether it also depends on the value of $\mathfrak{P}$ at other points. The discussion of § 4.3 depends on the assumption that the stress is uniform, and if we calculate the energy of interaction of the dislocation stresses and the applied stresses we find that the contribution from regions far from the dislocation line are not small. Eshelby (unpublished) has verified that in certain special cases (48) gives the correct result, and has also given a general proof (1951). Here we consider two examples.

The first verification is for the edge dislocation of § 4.41. If the stresses p_{rr} and $p_{r\theta}$ on the boundary of the cylinder are expanded in Fourier series, it is clear from eqns. (56) that only that component of P corresponding to a uniform p_{xy} shear stress does work when the dislocation moves.

For the eccentric screw dislocation of § 4.42, the work done is given by (60) as

$$W_e = \frac{b\delta\xi}{\pi} \int_{-\pi}^{\pi} \frac{V(R, \theta)\, d\theta}{R^2 - 2R\xi \cos \theta + \xi^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

where

$$V(R, \theta) = xyp_{xz} + y^2 p_{yz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

Now (§ 2.2), u_z is a harmonic function in the xy plane. It follows that p_{xz} and p_{yz} are harmonic. V is not harmonic, but the quantity $V'=V+\frac{1}{2}(R^2-x^2-y^2)p_{yz}$ is harmonic, and is equal to V on the boundary $x^2+y^2=R^2$.

We then have

$$W_e=\frac{b\delta\xi}{\pi}\int_{-\pi}^{\pi}\frac{V'(R,\theta)\,d\theta}{R^2-2R\xi\cos\theta+\xi^2},$$

which, by the use of Poisson's integral theorem for a plane harmonic function, may be written

$$W_e=2b\delta\xi V'(\xi,0)/(R^2-\xi^2).$$

Since $V(\xi, 0)=0$, this becomes

$$W_e=b\delta\xi p_{yz}(\xi,0). \qquad \text{(63)}$$

The force on the dislocation again depends only on the shear stress in its immediate neighbourhood.

4.6. FORCES BETWEEN DISLOCATIONS

The idea that dislocations exert forces on each other was introduced by Taylor, and forms the basis of his theory of work hardening (§ 5.31). He assumed that the force between two edge dislocations could be calculated from (48), where $\mathfrak{P}$ was the stress produced by one at the centre of the other. Koehler (1941) calculated the force between two edge dislocations of opposite sign, not necessarily lying in the same glide plane, by integrating the strain energy density over the whole body. Since the dislocations are of opposite sign, the strains are proportional to r^{-2} at large distances, and the energy is given by a convergent integral. His results differ from those of Taylor, and in particular he finds that the force is not proportional to the stress at the centre of each dislocation, although the numerical discrepancy is not very great. Cottrell (1949) considered a specimen in which one dislocation was already formed, and estimated the energy of interaction by considering the force required to make the second dislocation over a cut extending from its axis to a distant stress-free cylindrical surface with the first dislocation as its axis. The energy of interaction is the work done by the forces produced by the first dislocation acting through the displacements produced by the second dislocation. This method, which is essentially that of Volterra, has been extended by Bilby (1950 a). Cottrell found the force of interaction to be the same as that which would be produced by a uniform stress equal to the stress at the centre of each dislocation. Eshelby (unpublished) has verified this result.

The error in Koehler's calculation arises from the fact that the finite total energy is expressed as the difference between the infinite self energy of two dislocations and their infinite interaction energy. In evaluating these two energies the passage to infinity is made in the one case by considering a circle of large radius and in the other case by considering a strip of infinite length and large width. Eshelby has repeated the calculation using bipolar coordinates, which allows him to avoid infinite energies. He has also treated the problem in a manner which involves no infinite integrals and no infinite or discontinuous integrands by a generalization of the treatment given by Dean and Wilson (1947). The problem is treated as one of plane strain. An infinite sheet contains two equal circular holes joined by a cut along their line of centres. By forces applied *only* to the surfaces of the cut, these surfaces are given equal and opposite parallel displacements in an arbitrary direction in the plane, and the work done by these forces in producing the dislocation is calculated. If we take axes along and perpendicular to the arbitrary direction, the calculated energy is seen to be that of a pair of edge dislocations in arbitrary positions with equal and opposite Burgers vectors. The result again agrees with that obtained by Cottrell.

Since the total energy of a pair of dislocations of the same sign is infinite, it is difficult to calculate their interaction. We assume it is equal and opposite to the interaction of a pair of opposite sign.

4.61. *Force between Edge Dislocations*

The stress produced by an edge dislocation at the origin having a Burgers vector with Cartesian components $(b, 0)$ is given by eqns. (50) and (51). The force on a dislocation of opposite sign at the point (r, θ) is given by eqn. (45), and has components

$$\left.\begin{aligned} F_r &= -bDr^{-1} \\ F_\theta &= -bDr^{-1}\sin 2\theta. \end{aligned}\right\} \qquad (64)$$

The interaction energy may be obtained by integration, and is

$$W = bD\ln r - \tfrac{1}{2}bD\cos 2\theta, \qquad (65)$$

where $D = b\mu/2\pi(1-\sigma)$.

If the second dislocation cannot move out of its glide plane (fig. 19), the component of force producing conservative motion is

$$F_x = -bDr^{-1}\cos\theta\cos 2\theta = -bp_{xy}, \qquad (66)$$

or

$$F_x = -\tfrac{1}{4}bDy^{-1}\sin 4\theta. \qquad (67)$$

The symbol ⊥ represents an end view of an edge dislocation. The long stroke is the glide plane, and the short stroke represents the tip of the half plane of atoms which ends on the glide plane. The same dislocation is occasionally represented by the symbol ⊤.

The glide plane is a plane $y=\text{const.}$, and F_x is plotted as a function of x in fig. 20 (*a*). There is stable equilibrium for $\theta=\pi/4$ and unstable for $\theta=0$ and $\theta=\pi/2$, with symmetrical positions on the other side of the y axis.

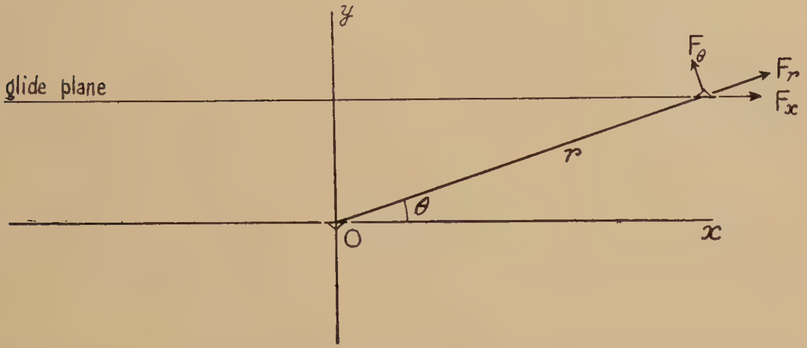
For two dislocations of the same sign, these forces are reversed. The position of stable equilibrium is given by $\theta=\frac{1}{2}\pi$ and the position of unstable equilibrium by $\theta=\frac{1}{4}\pi$. In this case the position of stable equilibrium does not have the lowest energy of all positions which can be reached by glide, for there is a long range repulsion between the dislocations. When $\theta=\frac{1}{2}\pi$, $r=y$, and when $\theta=\frac{1}{4}\pi$, $r=\sqrt{2}y$. Changing the signs in eqn. (65), we find

$$W(\theta=\tfrac{1}{2}\pi)=-bD\ln y-\tfrac{1}{2}bD$$

and

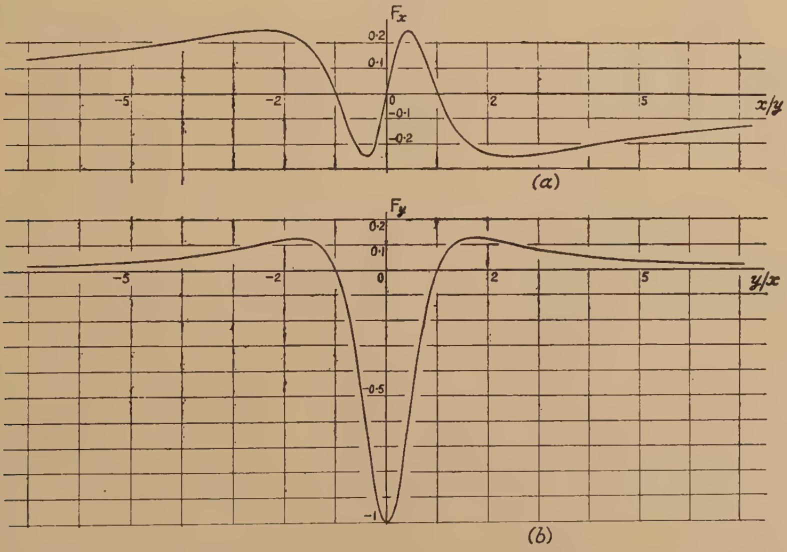
$$W(\theta=\tfrac{1}{4}\pi)=-bD\ln\sqrt{2}y.$$

Fig. 19



Force on a dislocation at (r,θ) produced by a dislocation of opposite sign at the origin. Only the component F_x is effective in causing glide. The dislocation at the origin would more usually be represented by the symbol $\perp$.

Fig. 20



(*a*) The component of force F_x acting in the glide direction of the dislocation of fig. 19. The unit of force is bD/y, where y is the (constant) distance of the glide plane from the dislocation at the origin. (*b*) The component of force F_y in the new glide direction if the moving dislocation of fig. 19 has its Burgers vector turned through a right angle while the dislocation at the origin is unaltered. The unit of force is bD/x.

The activation energy per unit length if the dislocation is to escape from its bound position is given by

$$W_a = bD(\tfrac{1}{2} - \tfrac{1}{2}\ln 2) = 0{\cdot}153\, bD. \qquad (68)$$

4.62. *Force between Screw Dislocations*

Consider two cylindrical holes of radius r_0, with axes a distance r apart. If a slit is made along a plane joining the axes, and the sides of the slit are displaced $\pm\frac{1}{2}b$ parallel to the axes, the result is a pair of screw dislocations. Since stresses round screw dislocations, like those round edge dislocations, vary as r^{-1}, we may use Eshelby's result that, in the limit $r_0 \ll r$, it is not necessary to annul the normal forces over the two cylinders, and the stress may be taken as a simple superposition of the stresses produced by each dislocation in a medium with no other singularities. With this approximation, the interaction energy is

$$W = (b^2\mu/2\pi)\ln r, \qquad (69)$$

and the force is

$$F_r = -b^2\mu/2\pi r. \qquad (70)$$

4.63. *Force between Parallel Dislocations with Burgers Vectors not Parallel*

It is obvious from the form of the function giving the energy density at any point that there is no interaction between edge and screw dislocations lying along parallel lines. It remains to consider two dislocations lying along lines parallel to Oz, and having Burgers vectors $(b, 0, 0)$ and $(0, b, 0)$. We may use the method of Cottrell and Bilby, considering the first dislocation already existing at the origin, and the second formed by displacements over the surface $R \geqslant r \geqslant r_0$, $\theta = \theta_0$. The energy of interaction is easily shown to be

$$W = \tfrac{1}{2}bD\sin 2\theta, \qquad (71)$$

and the force in the glide direction of the second dislocation is

$$F_y = -bDx^{-1}\cos^2\theta\cos 2\theta = bp_{yx}. \qquad (72)$$

When the integration is made over a radial cut, the angular dependence of W arises entirely from the stress function χ^1 of eqn. (52). The force F_y is plotted as a function of y in fig. 20 (b). There is stable equilibrium for $\theta = \frac{1}{2}\pi$ and for $\theta = -\frac{1}{4}\pi$, and unstable for $\theta = -\frac{1}{2}\pi$ and $\theta = \frac{1}{4}\pi$. The position of lowest energy is given by $\theta = -\frac{1}{4}\pi$; in this the zone of compression of the dislocation $(0, b, 0)$ overlaps the zone of dilatation of the dislocation $(b, 0, 0)$. The activation energy per unit length to move the dislocation from $y = \infty$ to $y = -x$ is

$$W_a = \tfrac{1}{2}bD. \qquad (73)$$

In general, if a dislocation lies along a line with direction cosines $\mathbf{l}$ we may resolve its Burgers vector $\mathbf{b}$ into an edge component $\mathbf{e}$ and a screw component s given by

$$\mathbf{e} = -\mathbf{l}\wedge(\mathbf{l}\wedge\mathbf{b})$$

and

$$s = \mathbf{l}\,\mathbf{b}.$$

If two dislocations lie along parallel lines, the shortest path between which is a vector $\mathbf{r}$, their interaction energy per unit length is given by

$$W=\frac{\mu}{2\pi}[s_1s_2\ln r+(1-\sigma)^{-1}(\mathbf{e}_1\mathbf{e}_2\ln r+\mathbf{e}_1\mathbf{r}\,.\,\mathbf{e}_2\mathbf{r}/r^2)]. \quad . \quad . \quad (74)$$

4.64. *Forces between Dislocations along Perpendicular Lines*

In §§ 4.61–4.63 we have considered the general interaction between dislocations lying on parallel lines. If the dislocation lines are perpendicular, the force between them is not constant along the length of each, but is concentrated in the region where they approach closely. If the lines are initially straight, their interaction may cause a bulge to appear in each where it approaches the other. In § 5.22 this difficulty is avoided by considering the interaction between an infinite grid of screw dislocations and a single screw dislocation lying parallel to the plane of the grid and perpendicular to each dislocation of the grid, and far from the grid. In the present section we shall neglect this complication, and shall also neglect the difficulties which arise on an atomic scale if one dislocation cuts another, so that the regions of misfit in their cores are superimposed through a small volume. The total interaction energy from this last cause is $cb^3\mu$, where c is a numerical coefficient probably of the order of 0·1 or 0·2, and the force is small unless the dislocations approach within say $3b$ of each other.

We may take the first dislocation to lie along the z axis, while the second lies parallel to the y axis and passes through the point $(X, 0, 0)$. Then the essentially different configurations which are possible are those listed in the following table:

Serial	Burgers vector of dislocation along [001]	Burgers vector of dislocation along [010]
1	[100] (edge)	[100] (edge)
2	[100] (edge)	[010] (screw)
3	[100] (edge)	[001] (edge)
4	[010] (edge)	[010] (screw)
5	[010] (edge)	[001] (edge)
6	[001] (screw)	[010] (screw)

4.641. *Two Edges, both Burgers Vectors along Shortest Distance between Lines.*—Although the energy density contains cross terms in p_{xx} and in the dilatation, there is no interaction between the dislocations, as may be seen by forming the second dislocation by a displacement over the cut $z=0$, $x>X$. This configuration occurs in *pencil glide* (Taylor and Elam 1926) and *cross slip* (Maddin *et al.* 1948).

4.642. *One Edge with Burgers Vector along Shortest Distance, One Screw.*—No interaction, by the argument of § 4.641. If the edge passes through the screw it develops a *jog* (see footnote to Mott 1951), so that it is still a pure edge, but no longer lies in one glide plane.

4.643. *One Edge with Burgers Vector along Shortest Distance, One Edge with Burgers Vector Parallel to Line of First.*—No force on second in its own slip direction, because the field of the first does not vary in this direction. Considering the first formed in the presence of the second by a displacement over the cut $y=0$, $x<0$, we find the interaction vanishes.

4.644. *One Edge with Burgers Vector Parallel to Line of Second, One Screw.*—No interaction, by the argument of § 4.641. The screw can pass through the edge, which then has a kink, but still lies entirely in one glide plane.

4.645. *Two Edges, each with Burgers Vector Parallel to Line of other.*—No force in glide direction of either, by the first argument of § 4.643. There is a finite interaction leading to non-conservative motion. We estimate this by considering the first dislocation to be formed in the cylinder $x^2+y^2<R^2$, $z^2<k^2R^2$, $R \gg X$, $k \gg 1$. The stresses are given in cylindrical coordinates by the stress functions (49) and (52), and the cylindrical surface is free from stress. There is an axial stress given by $p_{zz}=\lambda(p_{rr}+p_{\theta\theta})/2(\lambda+\mu)$, and a state of plane strain can only be maintained if normal stresses act over the ends of the cylinder $z=\pm kR$. Now let the second dislocation be made over the cut $z=0$, $x>X$. The energy of interaction is proportional to the integral of p_{zz} over this cut, which does not vanish, but contains a term linear in X leading to a constant force between the dislocations. During the formation of the second dislocation, the ends $z=\pm kR$ of the cylinder move. Since $k \gg 1$, they cannot distort but must move rigidly. The work done by the stresses acting on them therefore depends only on the total force and total couple on each end, each of which can easily be shown to vanish, so that the interaction depends only on the integral of p_{zz} over the cut.

4.646. *Crossed Screws.*—Finite interaction, which is discussed in § 5.22. Two screw dislocations cannot glide through one another, for after they have crossed each contains a jog which is an element of edge dislocation with its Burgers vector perpendicular to its direction of motion. Any further motion must be non-conservative.

4.647. *Summary of the Laws of Force between Dislocations.*—The results of §§ 4.61, 4.62, 4.63, 4.64 and 5.22 may be summarized in the following rules, which apply to dislocations on parallel or on perpendicular lines: two screw dislocations always cause each other to glide along the shortest distance between them. Two edge dislocations cause each other to glide if, and only if, their lines are parallel. If two perpendicular dislocations each have their Burgers vector parallel to the line of the other, there is an interaction which does not lead to glide. In all other cases there is no long range interaction.

4.65. *Dislocations in Thin Plates*

Eshelby and Stroh (1951) have considered dislocations in thin plates. The force between two screw dislocations lying normal to the plate falls off exponentially with the distance between them.

4.66. *Force between a Dislocation and a Solute Atom*

The interaction between dislocations and foreign atoms dissolved in the lattice forms the basis of the modern theory of the yield point which is observed in mild steel and other materials (Sutton and Willstrop 1937, Cottrell 1948, 1952 a, Nabarro 1948 a, Cottrell and Bilby 1949, Cottrell and Jaswon 1949, Bilby 1950 b). If the dissolved atom is regarded simply as a centre of dilatation occupying a volume δV greater than that of a normal atom (in the case of a substitutional solute) or causing an expansion δV (in the case of an interstitial solute), it does not interact with the shear stresses in the matrix, but only with the hydrostatic pressure P. A screw dislocation produces no hydrostatic pressure in regions where linear elastic theory is valid; the pressure produced by an edge dislocation is given by (50) as

$$\begin{aligned} P &= -p_{rr}-p_{\theta\theta}-p_{zz} \\ &= \frac{3\lambda+2\mu}{\lambda+2\mu}\cdot\frac{b\mu r^{-1}\sin\theta}{\pi}. \qquad (75) \end{aligned}$$

The energy of interaction is $p\delta V$. If one looks into the physical interpretation of this mathematical centre of dilatation (see references to § 4.3), it becomes clear that the problem needs to be defined in more detail. The dissolved atom must be regarded as a sphere having definite elastic properties. Changes in the assumptions alter the interaction energy by a numerical factor.

Frank, in discussion of Cottrell (1948), pointed out that in some cases, such as that of an interstitial solute atom in a body-centred cubic lattice, the solute also interacts with shear stresses in the matrix. In these cases there is also an interaction between solute atoms and screw dislocations, and these are the systems which show the most clearly marked yield phenomena. Crussard (1950) has given a quantitative theory of this interaction.

§ 5. ARRAYS OF DISLOCATIONS

We shall now discuss the properties of various arrays of dislocations. Most of the arrays contain dislocations uniformly spaced, but in § 5.1 we consider a more general case in which dislocations moving on the same glide plane reach equilibrium in a pattern which is not uniformly spaced.

5.1. A ROW OF DISLOCATIONS ON THE SAME GLIDE PLANE

Consider a set of n similar edge dislocations in the same glide plane, each subject to a force $\mathbf{f}$ tending to move it along the plane. Suppose the dislocation lying furthest in the $\mathbf{f}$ direction is prevented by some obstruction from moving across the glide plane. Cottrell (1949) has pointed out that the force per unit length acting on the obstruction is $n\mathbf{f}$. For if the obstruction moves a distance d in the direction of $\mathbf{f}$, each of the n dislocations moves the same distance, and the work done by the external forces is nfd. Thus a glide plane containing mobile dislocations produces stress concentrations analogous to those produced by a crack.

Taylor (cf. Starr 1928) and Zener (1948 a) have suggested that such a glide plane may have no resistance to shear, so that the stress concentration at its end is actually equal to the stress concentration produced by a crack. For a crack of length L, with a radius of curvature ρ at the end, this stress concentration has a maximum value of $(2L/27\rho)^{1/2}$, at a point a short distance beyond the tip of the crack (Starr).

Consider a glide plane of width L, and suppose that slip begins at one end and cannot continue right across because the glide planes of the neighbouring crystal are inclined to those of the first crystal. The repulsion between similar dislocations means that under a given applied stress P only a definite number $n(P)$ of dislocations will enter the glide plane. The force acting on unit length of the leading dislocation is then $bPn(P)$. Since $n(P)$ is clearly an increasing function of P, the stress on the leading dislocation increases more rapidly than P, whereas the concentrated stress produced by a crack is proportional to the applied stress. This problem has been treated by Eshelby, Frank and Nabarro (1951), who find that, to a close approximation,

$$n(P)=PL/2D, \quad \dots\dots\dots \quad (76)$$

where D is given by (4). This is the stress concentration factor K_d. We may compare it with the factor for a crack $K_c=(2L/27\rho)^{1/2}$ by putting $\rho=b$. We find

$$\frac{K_d}{K_c}=\left(\frac{27L}{2b}\right)^{1/2}\pi(1-\sigma)\frac{P}{\mu}, \quad \dots\dots\dots \quad (77)$$

which is equal to unity when

$$\frac{P}{\mu}=\frac{1}{\pi(1-\sigma)}\left(\frac{2b}{27L}\right)^{1/2}. \quad \dots\dots\dots \quad (78)$$

The concentrated stress is then given by

$$\frac{P}{\mu}=\frac{2}{27\pi(1-\sigma)}\bumpeq 0{\cdot}035. \quad \dots\dots\dots \quad (79)$$

If the concentrated stress is less than this, $K_d<K_c$, while for higher stresses $K_d>K_c$.

For some purposes the important quantity is not the stress on the leading dislocation, but the stress distribution in a fairly wide region ahead of the tip of the glide plane. It can be shown that, in a region ahead of the leading dislocation which is a distance more than $D^2/4LP^2$ and less than L from the leading dislocation, the stress is nearly equal to that which would be produced by a crack of length $4L$. Within this distance $D^2/4LP^2$ of the leading dislocation, the stress of the leading dislocation itself predominates, while at large distances the stress exceeds P by the stress field of a group of n dislocations at the origin. The relative displacement of the two sides of the glide plane near its tip oscillates around the value of the relative displacement of the sides a crack of length $4L$ under the same stress. Koehler (1952) has estimated the tensile stresses near the leading dislocation.

The method used to obtain (76) depends on a result of Stieltjes (1885): its application to this problem was suggested by H. Heilbronn.

Suppose the leading dislocation is held up at the point $x_1=0$, and the remaining dislocations are in equilibrium at points $x_2 \ldots . x_n$ such that $x_1<x_2< \ldots . <x_n$. Then the ith dislocation is acted on by a force $f=-Pb$ tending to reduce its coordinate, and by a force of interaction $bD/(x_i-x_j)$ with each other dislocation j. The equilibrium values of x_j are determined by the set of equations

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{bD}{x_i-x_j}=bP, \ldots . i=2, 3 \ldots . n, \qquad (80)$$

and we require the largest value of n for which $x_n<L$.

Put

$$F(x)=(x-x_1)(x-x_2) \ldots . (x-x_n), \qquad (81)$$

so that the shear stress produced by all the dislocation at any point x on the glide plane is $DF'(x)/F(x)$. Put also

$$g_i(x)=\frac{F'(x)}{F(x)}-\frac{1}{x-x_i}=\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{x-x_j}. \qquad (82)$$

Then eqns. (80) may be written

$$g_i(x_i)=a,$$

where

$$a=P/D=2\pi(1-\sigma)P/b\mu. \qquad (83)$$

As x tends to x_i, so $g_i(x)$ tends to $\frac{1}{2}F''(x_i)/F'(x_i)$. The problem is thus reduced to the determination of a polynomial of degree n which vanishes at $x=0$ and at $n-1$ distinct positive values $x_2 \ldots . x_n$, and for which $\frac{1}{2}F''(x_i)/F'(x_i)=a$ at each zero. The polynomial will certainly satisfy this condition if it satisfies the differential equation

$$\tfrac{1}{2}F''-aF'+nax^{-1}F=0, \qquad (84)$$

and it is easily shown that this differential equation has in fact a polynomial solution of degree n satisfying the required conditions. For small x, the second term in (84) may be neglected, and the equation solved to show that the first ν dislocations, where ν is small compared with n, are contained in a distance proportional to ν^2/na. The total number n of dislocations occupies a width x_n of the glide plane, and it will be shown that for large n the value of x_n is given very closely by

$$x_n=2n/a. \qquad (85)$$

Equating this to L, we find that the maximum number of dislocations which can be held in the glide plane is given by (76).

To justify (85), we proceed as follows. First suppose that all the dislocations except the last are constrained to lie at the origin. The force they produce on the last dislocation is $(n-1)b^2\mu/X_n$, where X_n is the equilibrium position of the last dislocation in this constrained system.

Equating this to Pb, we have $X_n=(n-1)/a$. But it is clear that if all dislocations but the first are now released, all will move away from the first, and so $x_n>X_n=(n-1)/a$. Heilbronn has obtained the closer result that $x_n>(2n-\gamma n^{1/3})/a$, where γ is an absolute constant. The determination of a lower limit to x_n is based on the differential equation (84). We write

$$F(x)=ue^{ax},$$

and (84) becomes

$$u''+(2nax^{-1}-a^2)u=0. \quad \ldots \ldots \quad (86)$$

Suppose $x_n \geqslant 2n/a$, and choose the sign of the leading term in the polynomial solution of (84) so that F and u are positive when $x>x_n$. Then $u'(x_n)>0$, and $u''(x)>0$ for $x>x_n$. Integrating, $u'(x)>u'(x_n)$, and, integrating again, $u(x)>(x-x_n)u'(x_n)$ for $x>x_n$. But in the polynomial solution of (84), $F(x)=O(x^n)$ as $x\to\infty$, and so $u(x)\to 0$ as $x\to\infty$. This is a contradiction, and the assumption $x_n \geqslant 2n/a$ is false. Hence $2n-\gamma n^{1/3}<ax_n<2n$.

A further case of interest is that in which no external stress is applied, and the first and last dislocations are held by obstructions. This problem and a number of others were discussed by Eshelby *et al.* Nye (1949), in experiments on crystals of silver chloride, met a related problem, in which the dislocations were held up not by internal obstacles in an effectively infinite medium, but by the surface of the crystal. If the glide plane $y=0$ is bounded by the edges of the crystal $x=\pm L$, the problem must be solved with the additional condition that the stresses p_{xx} and p_{xy} are zero on the surfaces $x=\pm L$. The solution was obtained by Mann on the assumption that dislocations are distributed at equal spacings across the glide plane. In all of these problems the glide plane becomes bent in the way shown in fig. 2. It does not seem possible to extend the method of Stieltjes to the case of a slip plane containing both positive and negative dislocations. Following Leibfried (1951), we may consider the limiting form of the distribution when the total number of dislocations is very large. We suppose that the number of dislocations lying between x' and $x'+dx'$ on the glide plane is $\mathscr{D}(x')\,dx'$. At the point x these produce a stress

$$\int_{-\infty}^{\infty}\frac{D\mathscr{D}(x')\,dx'}{x-x'}, \quad \ldots \ldots \quad (87)$$

where the Cauchy principal value of the integral is understood. The condition of equilibrium of a dislocation at the point x is thus

$$\int_{-\infty}^{\infty}\frac{\mathscr{D}(x')\,dx'}{x-x'}=-\frac{P}{D}, \quad \ldots \ldots \quad (88)$$

where P is the external stress. This equation must hold at all points for which $\mathscr{D}(x)$ is not zero. The derivation of (88) is purely formal, for the stress (87) has a singularity at each dislocation in the distribution, and these are the only points for which (88) should be true. However, comparison of (88) with § 2.1 of Part III shows that (88) is the condition that the glide plane containing dislocations should have no resistance to shear, and the analysis of Eshelby *et al.* shows that, in a particular case, this is correct.

Using (88), Leibfried has solved a number of problems, including all those previously treated by Eshelby *et al.* He has also treated the case of a slip plane of length $2a$ containing equal numbers of positive and negative dislocations, forming a model of a slip plane containing a Frank–Read source. The solution of (88) for this problem is

$$\mathscr{D}(x) = -Px/\pi D(a^2-x^2)^{1/2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

The total number of dislocations of either sign is

$$\int_{-a}^{0} D(x)\,dx = Pa/\pi D, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

and the shear strain is proportional to

$$b\int_{-a}^{a} x\mathscr{D}(x)\,dx = bPa^2/2D = \pi(1-\sigma)a^2P/\mu, \quad . \quad . \quad . \quad (91)$$

which is proportional to the applied stress. Segedin's theory (1951) of a penny-shaped crack in a sheared elastic body may be interpreted similarly.

Leibfried also considered a glide plane containing less than $Pa/\pi D$ dislocations of each sign, representing a Frank–Read source which has become blocked before producing the maximum number of loops which the glide plane can accommodate under the applied stress.

5.2. THE GRAIN BOUNDARY AS A WALL OF DISLOCATIONS

5.21. *A Wall of Edge Dislocations*

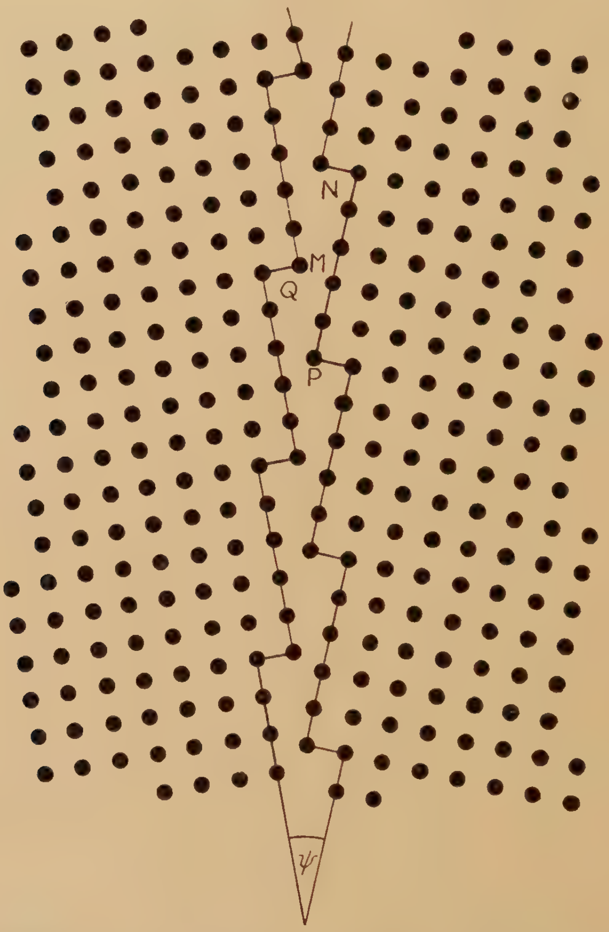
The first calculation of the stresses produced by joining two crystals whose axes are inclined to one another at a small angle is due to Taylor (1934). He considered two simple cubic lattices with lattice spacing b, inclined to one another at a small angle ψ, so that the z axes in the two crystals are parallel, while the surface of separation is parallel to this axis and bisects the angle between the crystals. We call this a *tilt boundary*. Take the y axis in the boundary surface. Then before the crystals are joined together they have the form shown in fig. 21, where a step occurs after every $2/\psi$ atoms. The join is made with least energy if the step M on the left is brought near to the gap N on the right, when the step P on the right falls in the gap Q on the left. If the opposing surfaces are now pressed together, each suffers a saw-tooth deformation of amplitude $\pm\frac{1}{4}b$, and of wavelength b/ψ.

By a Fourier analysis of this deformation it is easily shown that the stress P at a point distant x from the surface of misfit is of order

$$P = 2\mu\psi\eta\,\mathrm{e}^{-\eta}, \text{ where } \eta = 2\pi\psi\,|\,x\,|/b. \quad . \quad . \quad . \quad . \quad (92)$$

The stress is only appreciable at distances from the surface of misfit less than about $2b/\psi$, which is the distance between successive steps in the face of a crystal. Taylor suggested that these regions of high stress were the boundaries of the mosaic blocks which were postulated by Darwin to account for the intensity of x-ray reflections, and later identified with various microscopically observed structures in crystals.

Fig. 21



Two grains inclined at a small angle ψ. They join with least energy if the corners P and M fit in the gaps Q and N.

Dislocations near the middle of a mosaic block can move under an external stress equal to that given by (92), where the value of η is that for the middle of a block, and this should be the elastic limit. Taking the width of a block as $L=2x=10^{-4}$ cm, $b=3\times10^{-8}$ cm, $\psi=3\times10^{-4}$ radians, and $\mu=2{\cdot}6\times10^{11}$ dyne cm^{-2}, the calculated elastic limit becomes

260 g. wt. mm^{-2}, which is a reasonable order of magnitude. Taylor assumed that dislocations could not cross the mosaic boundaries at low temperatures, but could penetrate regions of higher and higher stress in the boundaries as the temperature was raised.

Burgers (1939 a) pointed out that the composite crystal could be regarded as a single crystal containing edge dislocations of vector $(0\, b,\, 0)$, spaced at regular intervals b/ψ along the y axis. The value of η for the middle of a mosaic block is π times the number of dislocations in a wall of the block. This point of view makes obvious a result which is implicit in Taylor's analysis, that the stress is concentrated along a row of parallel lines in the xz plane, and falls off rapidly between these lines. Bragg (1940) has also considered boundaries of this type.

It is easily verified that if a crystal contains the infinite row of dislocations specified in the previous paragraph, the regions far from the x plane and lying on either side of it are unstrained, and rotated through angles $\pm\frac{1}{2}\psi$ about the z axis. At these distant points the uniform row of dislocations may be regarded as a continuous distribution of dislocation spread over the x plane with density ψ/b. The stresses and displacements Φ may be obtained from those ϕ given in § 2.1 by integration, using the formula

$$\Phi = \lim_{L\to\infty} \frac{\psi}{b}\int_{-L}^{L} \phi\, dy. \qquad (93)$$

The stresses p_{xx} and p_{yy} are odd in y, and their integrals vanish. The integral of p_{xy} converges to zero as $L\to\infty$, and the crystal is thus free from stress. The rotation produced by a single dislocation is given by

$$\varpi = -\frac{b}{2\pi}\frac{x}{x^2+y^2}.$$

Substituting in (93), the rotation produced by a sheet of dislocations is found to be $-\frac{1}{2}\psi \operatorname{sgn} x$, where $\operatorname{sgn} x$ is 1 if $x>0$ and -1 if $x<0$.

It is easy to make a rough estimate of the energy of such a wall of dislocations (Shockley and Read 1949), by using the formulae of § 4.41 in conjunction with the result that the stress is confined to a strip of width roughly equal to the distance between dislocations. We regard the total energy as the sum of equal contributions from each dislocation, this contribution arising from a circular cylinder of radius $b/2\psi$ with the dislocation lying along its axis. Within this cylinder the stress is approximately that produced by a single dislocation in an infinite medium. Since $p_{rr}=p_{\theta\theta}$, the energy density is $\frac{1}{2}\mu^{-1}p_{r\theta}{}^2+\frac{1}{2}(\lambda+\mu)^{-1}p_{rr}{}^2$. If this is integrated from $b/2\psi$ right down to $r=0$, the result is infinite, and we must cut the integration off at some radius r_0, of the order of b. We expect that r_0 will be independent of ψ, provided that ψ is reasonably small, that is to say, that the energy in the core of a dislocation is little affected by the presence of other dislocations more than a few times b away from it. A discussion of this assumption and an estimate of r_0 depend on assumptions about the atomic forces in the core (§ 2.4 of Part III).

The energy associated with unit length of each dislocation is then

$$\frac{b^2\mu}{4\pi(1-\sigma)}\ln\frac{b}{2r_0\psi},$$

which we may write in the form

$$\frac{b^2\mu}{4\pi(1-\sigma)}\ln\frac{\psi_0}{\psi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

with

$$\psi_0=b/2r_0.$$

The distance between dislocations in the wall is b/ψ, and the energy of unit area of the wall is thus

$$\frac{b\mu}{4\pi(1-\sigma)}\psi\ln\frac{\psi_0}{\psi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

A more detailed discussion of the energy of a grain boundary is given in § 5.23.

The formation of a wall such as that shown in fig. 22 in a region containing a number of dislocations of the same kind does not require an activation energy. Any two dislocations arranged as in fig. 19 with the angle θ lying between $\pi/4$ and $3\pi/4$ will move until the line joining them is perpendicular to their glide plane. Suppose that the process of building up a wall has continued until the wall of fig. 22 has been formed, which contains dislocations at a uniform spacing of w stretching from $(0, L)$ to $(0, -L)$. This is the process of polygonization observed by Lacombe and Beaujard (1947), Lacombe (1948), Cahn (1948, 1949, 1950), Guinier and Tennevin (1948, 1950) and Dunn and Daniels (1951). The theory was first discussed by Cottrell (1949). The force in the glide direction on a dislocation at a point (x, y) at a distance from the wall large compared with w is given by eqn. (67) with sign reversed as

$$F_x=\tfrac{1}{4}bD\int_{-L}^{L}\frac{\sin 4\theta}{y-\eta}\frac{d\eta}{w},$$

where $\tan\theta=(y-\eta)/x.$

This reduces to

$$F_x=\frac{2LbD}{w}\,\frac{x(x^2-y^2+L^2)}{(x^2-y^2+L^2)^2+4x^2y^2}. \quad . \quad . \quad . \quad (96)$$

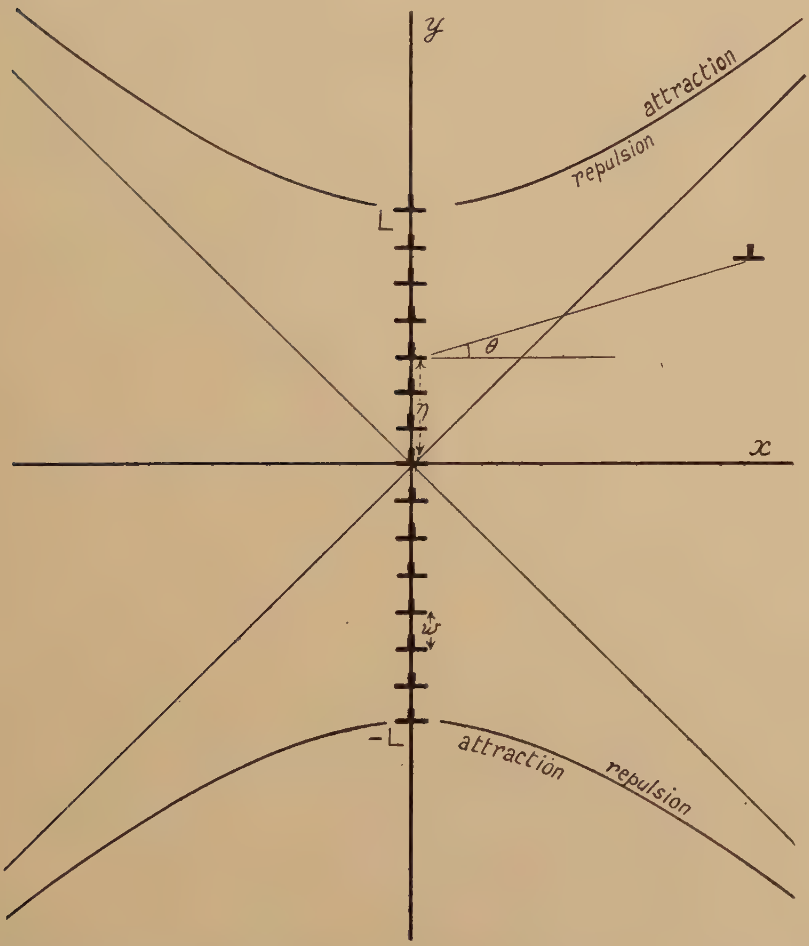
The denominator of this expression is always positive (except at the ends of the wall), and the force tends to pull the dislocation into the plane of the wall or to drive it out according as $(x^2-y^2+L^2)$ is positive or negative. The region of attraction is bounded by the hyperbola sketched in the figure.

When the dislocation comes within a distance w of the wall, it is no longer justifiable to represent the sum over all the dislocations in the wall by an integral. If its glide plane coincides with the glide plane of a dislocation already in the wall, the position of equilibrium will clearly be one in which neither dislocation lies exactly in the plane $x=0$, while a dislocation gliding midway between two dislocations in the wall may reach this plane. We shall consider in detail the force on a dislocation of an infinite wall if it glides out of the wall. The force when it is at the point $(x, 0)$ is

$$F_x = \tfrac{1}{2} bD \sum_{n=1}^{\infty} \frac{\sin 4\theta}{nw},$$

where $\tan\theta = nw/x.$

Fig. 22



Polygonization. A wall of dislocations stretching from $y=-L$ to $y=L$ exerts a long range attraction sweeping into the line of the wall dislocations above and below the two branches of the hyperbola. The dislocation shown, which lies between the branches, is repelled.

This may be written

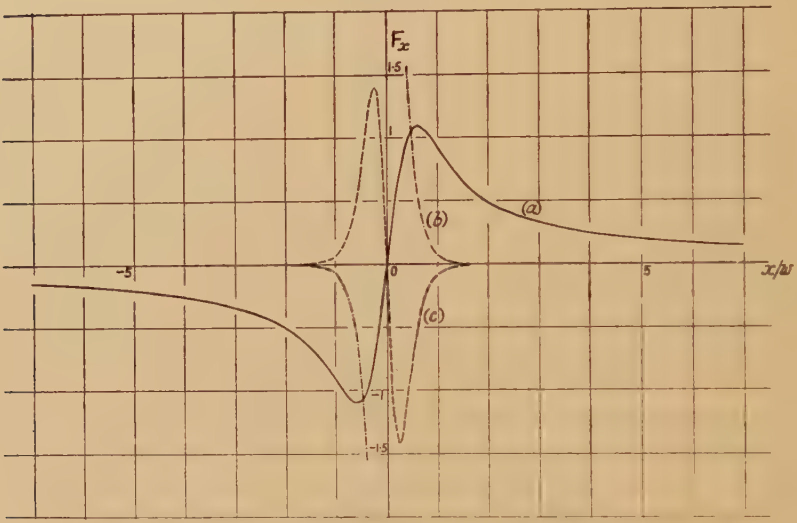
$$F_x = \frac{2bD}{x} \sum_{n=1}^{\infty} \frac{1 - n^2w^2/x^2}{(1+n^2w^2/x^2)^2}$$

$$= -\frac{bD}{x}\left[1 - \frac{\pi^2 x^2}{w^2} \bigg/ \sinh^2 \frac{\pi x}{w}\right]. \quad \ldots\ldots \quad (97)$$

For x small compared to w the expression in square brackets tends to $-\pi^2x^2/3w^2$, and the restoring force is $-\pi^2bDx/3w^2$, while for large x the expression tends exponentially to -1, and the force tends to $-bD/x$. The force is plotted in fig. 23 (a).

The force on a dislocation moving in the presence of an infinite regular wall along the glide plane of one of the dislocations in the wall is obtained by adding the repulsion $F_x=bD/x$ of a single dislocation at the origin. It tends exponentially to zero for large x in accordance with Taylor's formula (92) for the stress. The last formula in Burgers's (1939 a) paper gives the stress p_{xy} produced by a complete wall at any point, and the force on a dislocation is bp_{xy}. For a dislocation moving on the glide plane of a dislocation in the wall the force is

Fig. 23



Force on a dislocation approaching a wall of dislocations spaced at equal distances w. The unit of force is bD/w. (a) There is one dislocation missing from the wall, and the mobile dislocation is moving in this glide plane. (The sign of F_x should be reversed.) (b) Dislocation approaching a complete wall along the glide plane of a dislocation in the wall. (c) Dislocation striking the wall midway between two dislocations.

$$F_x=\frac{bD}{x}\frac{\pi^2x^2}{w^2}\Big/\sinh^2\frac{\pi x}{w}, \qquad (98)$$

in agreement with eqn. (97), while in the case of a dislocation which would strike the wall midway between two dislocations the force is

$$F_x=-\frac{bD}{x}\frac{\pi^2x^2}{w^2}\Big/\cosh^2\frac{\pi x}{w}. \qquad (99)$$

In the second case the dislocation is attracted into the wall. Its binding energy when it reaches the wall is

$$W=bD\ln 2. \qquad \text{(100)}$$

The forces are shown in curves (b) and (c) of fig. 23. Seeger (1951) has evaluated the forces on dislocations lying on intermediate glide planes.

5.22. *A Wall of Screw Dislocations*

There is a radial repulsion between two similar edge dislocations, but, because the interaction energy of a pair of edge dislocations depends on the direction of the line joining them, and because they can move only in parallel glide planes, it is possible for their interaction to lead to a diminution of the distance between them. Neither of these factors applies to the interaction between two screw dislocations, and they will always repel one another. Nevertheless, it is possible to form walls of screw dislocations which are stable provided that the dislocations at the boundaries of the wall are held fixed. These walls consist of grids of screw dislocations lying along two perpendicular directions (Frank 1948, van der Merwe 1950), and may be called *twist boundaries.*

We begin by calculating the effect at distant points of an infinite row of screw dislocations having Burgers vectors $(0,\ 0,\ b)$, spaced at equal distances w along the y axis. From eqns. (14) and (93) we see that the only stress which does not vanish is

$$p_{yz}=\frac{b\mu}{2\pi w}\int_{-\infty}^{\infty}\frac{x\,dy}{x^2+y^2}=\frac{b\mu}{2w}\operatorname{sgn} x. \qquad \text{(101)}$$

The rotation is

$$\varpi_x=\frac{b}{4w}\operatorname{sgn} x. \qquad \text{(102)}$$

Now consider a screw dislocation $(0,\ b,\ 0)$, passing through the point $(X,\ 0,\ 0)$. The stress (84) exerts on this a force per unit length $F_x=-b^2\mu/2w\operatorname{sgn} X$, which pulls it towards the wall. If, instead of a single screw dislocation $(0,\ b,\ 0)$, we have a sheet of similar dislocations spaced w apart in the plane $x=$const., the force between the two sheets is an attraction of $b^2\mu/2w^2$ per unit area. If the two sheets coincide, a single dislocation of the second sheet which breaks away and reaches the point $(X,\ 0,\ 0)$ is subject to a restoring force

$$\begin{aligned}F_x&=-\frac{b^2\mu}{2w}\left(1-\frac{2w}{\pi}\sum_{1}^{\infty}\frac{X}{X^2+n^2w^2}\right)\\&=-\frac{b^2\mu}{2w}\left[1-\coth\frac{\pi X}{w}+\left(\frac{\pi X}{w}\right)^{-1}\right]. \qquad \text{(103)}\end{aligned}$$

The function in square brackets decreases from 1 when $X=0$ to 0 when $X=\infty$.

If the whole second sheet is in the plane $x=X$, it produces a stress

$$p_{yz}=-\frac{b\mu}{2w}\operatorname{sgn}(x-X)$$

and a rotation

$$\varpi_x=\frac{b}{4w}\operatorname{sgn}(x-X).$$

When $X=0$, the stress cancels that given by (101), so that a wall of crossed screws produces no stress at large distances, while the total rotation is $\varpi_x=b/2w\operatorname{sgn} x$. Rotations of this kind have been observed in plastically deformed crystals by Heidenreich and Shockley (1948) and Wilman (1950, 1951). van der Merwe has illustrated the formation of such a wall of crossed screws by considering a crystal split along a plane in which the atoms lie in a simple square array. One half of the crystal is rotated with respect to the other, and the cut surfaces are again placed in contact. The resulting configuration is shown in fig. 24 (*a*). The forces acting across the boundary tend to pull neighbouring rows of atoms on the two sides into alignment, which leads to the configuration of fig. 24 (*b*), in which the grid of screw dislocations is indicated.

5.23. *The General Grain Boundary as a Wall of Dislocations*

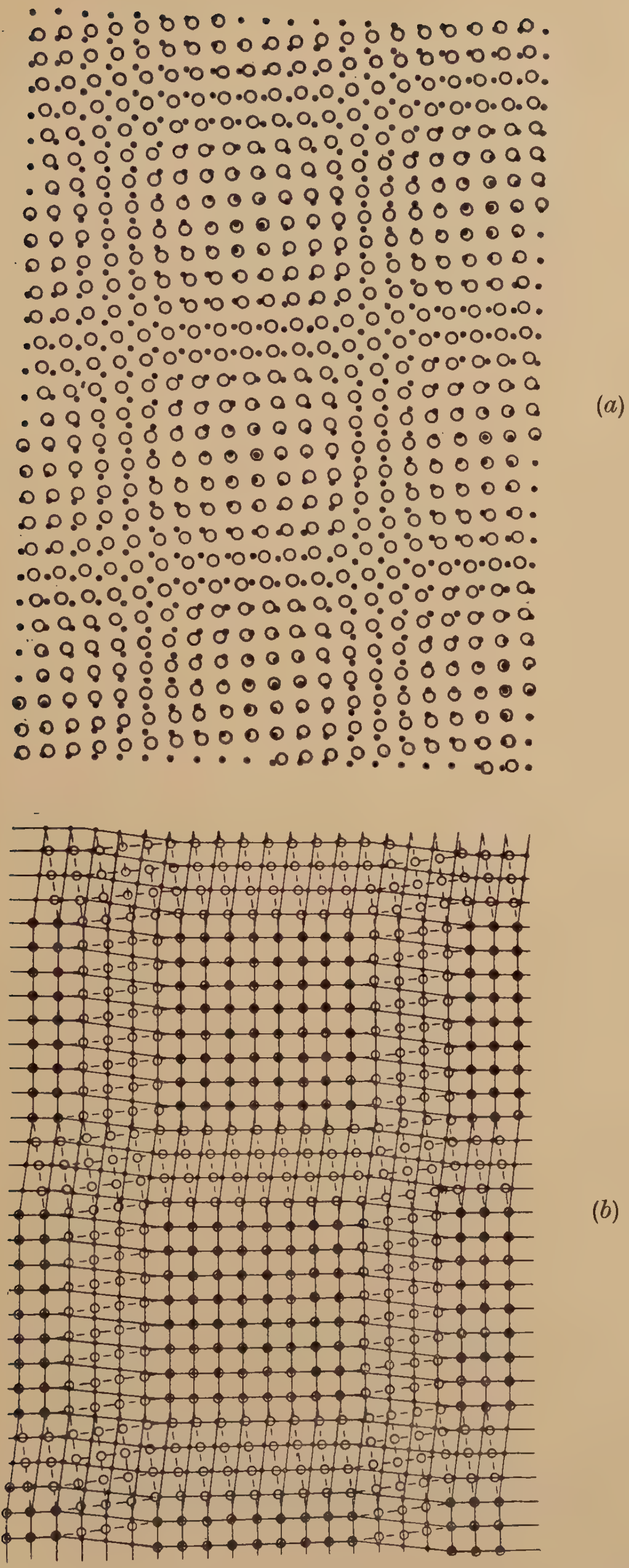
As an example of a boundary containing edge dislocations of two kinds, we may consider fig. 25. It follows from the discussion of § 4.63 that this array of dislocations is in stable equilibrium, and it produces a stress field tending exponentially to zero at distances greater than w from the wall. The lattices on either side of the wall are rotated through angles of $\pm b/2\sqrt{2}w$, and this mixed wall thus represents a boundary lying unsymmetrically between two grains.

A more general case is shown in fig. 26. We use the notation of Read and Shockley (1950), but write ψ where they write θ. The line OP, inclined at an angle ϕ to the x axis, forms the boundary between portions of the lattice which are rotated through small angles of $\pm\frac{1}{2}\psi$. Unit length of OP intercepts $b^{-1}\sin(\phi-\frac{1}{2}\psi)$ x-planes and $b^{-1}\cos(\phi-\frac{1}{2}\psi)$ y-planes of the lattice on the left, and $b^{-1}\sin(\phi+\frac{1}{2}\psi)$ x-planes and $b^{-1}\cos(\phi+\frac{1}{2}\psi)$ y-planes of the lattice on the right. The total number of planes intercepted is, for small ψ, $b^{-1}\psi\cos\phi$ x-planes and $b^{-1}\psi\sin\phi$ y-planes, corresponding to $b^{-1}\psi\cos\phi$ dislocations of type $(b, 0, 0)$ and $b^{-1}\psi\sin\phi$ dislocations of type $(0, b, 0)$ in unit length of the wall.

Shockley and Read (1949) assumed that each etch pit in the sub-boundaries observed by Lacombe (1948) marks the emergence of a single dislocation. They compared the spacing of etch pits with that calculated from the observed difference of orientation between neighbouring sub-grains, and found satisfactory agreement. (See also a footnote to Read and Shockley 1950.)

Frank (1950 b) has shown how the boundary between two grains may be analysed into dislocations when the angle ψ between the grains is not small, and the axis of their relative rotation meets the boundary

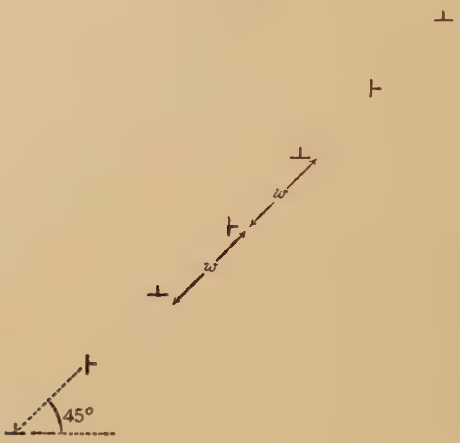
Fig. 24



Resolution of a twist boundary into a grid of crossed screw dislocations. (*a*) Two planes of atoms rotated by a small angle about their common normal, and superimposed. (*b*) (after W. T. Read.) Small displacements of the atoms allow the lattices to fit well over large areas, separated by screw dislocations.

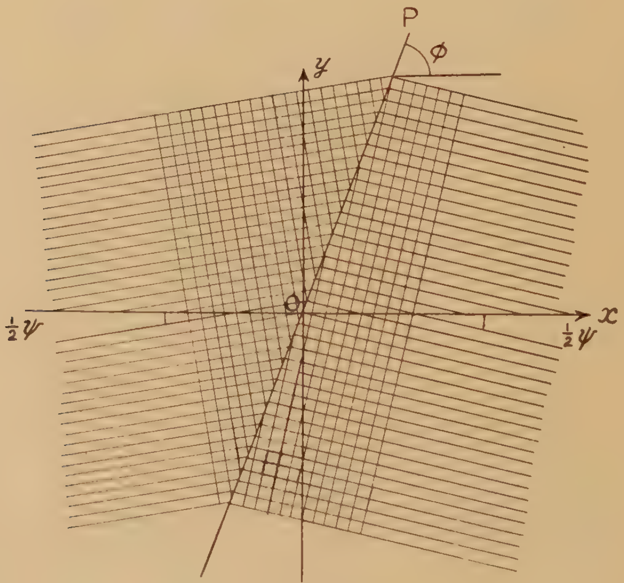
obliquely. He introduced the idea of the median lattice 𝔄 from which the lattices 𝔅 and ℭ of the two grains may be derived by equal and opposite rotations, and considered 𝔅 and ℭ to be derived from 𝔄 by inserting dislocations characteristic of the lattice 𝔄 along the grain

Fig. 25



A stable wall lying at 45° to each glide direction, with two types of dislocation equally spaced.

Fig. 26



A general tilt boundary. The two crystals are rotated by angles $\pm\frac{1}{2}\psi$, and join along a line making an angle ϕ with the x axis.

boundary. Since the rotation need not be small, the characteristic dislocations of $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are different. He defined corresponding points B and C in the two lattices to be points which are derived by rotation from the same point A of the original lattice. Take the origin of coordinates O in the boundary, and consider a point A which in the original lattice had coordinates $\mathbf{r}$. Let the lattices $\mathfrak{B}$ and $\mathfrak{C}$ be derived from $\mathfrak{A}$ by rotations about a line $\mathbf{l}$ through the origin through angles $\pm\frac{1}{2}\psi$. The coordinates of the point B referred to the lattice $\mathfrak{A}$ are

$$\mathbf{r}_B=\mathbf{r}+\mathbf{l}\wedge\mathbf{r}\sin\tfrac{1}{2}\psi+\mathbf{l}\wedge(\mathbf{l}\wedge\mathbf{r})(1-\cos\tfrac{1}{2}\psi),$$

while those of C referred to the same lattice are

$$\mathbf{r}_C=\mathbf{r}-\mathbf{l}\wedge\mathbf{r}\sin\tfrac{1}{2}\psi+\mathbf{l}\wedge(\mathbf{l}\wedge\mathbf{r})(1-\cos\tfrac{1}{2}\psi).$$

The separation of the corresponding points B and C is thus $BC=2\mathbf{r}\wedge\mathbf{l}\sin\frac{1}{2}\psi$. This is a dislocation composed of the last three of Volterra's orders of dislocation (§ 3.2), but it is convenient to regard it as a continuous uniform distribution of dislocations of the first three orders. The points which, before the dislocation was made, coincided at $\mathbf{r}$, are now separated by $2\mathbf{r}\wedge\mathbf{l}\sin\frac{1}{2}\psi$, while neighbouring points at $\mathbf{r}+d\mathbf{r}$ are now separated by $2(\mathbf{r}+d\mathbf{r})\wedge\mathbf{l}\sin\frac{1}{2}\psi$. The dislocation introduced across the grain boundary between the points $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$ is $2d\mathbf{r}\wedge\mathbf{l}\sin\frac{1}{2}\psi$. We may introduce the vector $\mathbf{s}=2\mathbf{l}\sin\frac{1}{2}\psi$, which defines the axis and amount of the relative rotation of the two lattices, and is independent of the position of their boundary. Then the density of dislocations in any part of their boundary is defined by the statement that a line $\mathbf{r}$ in the boundary crosses dislocations of total strength $\mathbf{r}\wedge\mathbf{s}$.

Read and Shockley have given a detailed analysis of the energy of a grain boundary of the type shown in fig. 26, and Read (1952) has considered a general grain boundary. The calculated dependence of the energy on ψ and ϕ agrees well with the measurements of Dunn and Lionetti (1949) and Dunn, Daniels and Bolton (1950). A review of the experimental results has been given by Fisher and Dunn (1952). The energy is most conveniently calculated by integrating the tangential stress across the slip plane of each dislocation from infinity to the actual position of the dislocation (§ 4.6). It is given by

$$E=E_0\psi(A-\ln\psi), \qquad (104)$$

where

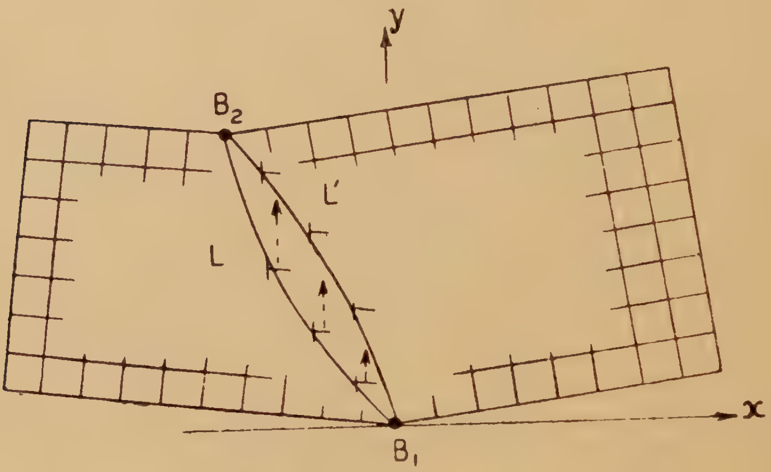
$$E_0=\frac{b\mu}{4\pi(1-\sigma)}(\sin\phi+\cos\phi) \qquad (105)$$

and

$$(\sin\phi+\cos\phi)A=(\sin\phi+\cos\phi)A_0-\sin\phi\cos\phi\,(\sin\phi+\cos\phi)$$
$$-(\sin\phi\ln\sin\phi+\cos\phi\ln\cos\phi). \qquad (106)$$

The constant A_0 corresponds to the constant $\ln\psi_0$ of (94). It is a measure of the energy in the core of a dislocation, and occurs with a factor $\psi(\sin\phi+\cos\phi)$, which measures the total number of dislocations in unit length of the wall. The value of $A-A_0$ given by (106) varies between $-0{\cdot}15$ and $+0{\cdot}13$. Since the estimated values of A_0 lie between 0·5 and 0·8, the variation of A with ϕ may usually be neglected. (The validity of this argument is now in doubt. See § 2.42 of Part III.) If we neglect the variation of A with ϕ, (104) and (105) show that the two boundaries B_1LB_2 and $B_1L'B_2$ of fig. 27 have the same energy. Read and Shockley also show that a movement of all the dislocations along their glide planes from one boundary to the other does not produce a relative displacement of the two sides of the crystal, provided the angle ψ is small. A relative displacement is produced only if the dislocations move non-conservatively by diffusion.

Fig. 27



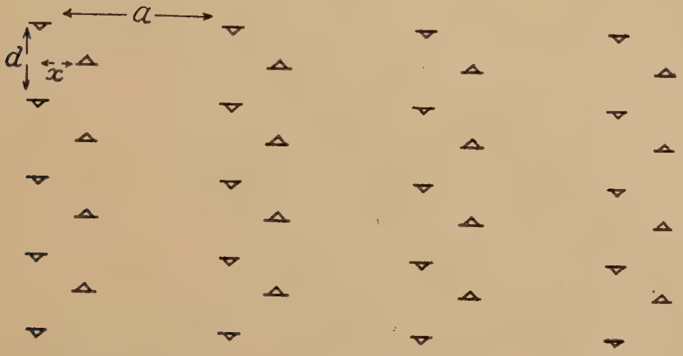
A small angle tilt boundary (after Read and Shockley). The y dislocations are not shown. The wall can move by glide alone from B_1LB_2 to $B_1L'B_2$.

If the angle ψ is not small, other considerations arise. Sometimes more than one pair of angles (ψ, ϕ) will provide a reasonable description of the same boundary. The analysis of particular cases shows that the predicted properties of such a boundary are independent of the description employed. Moreover, we have tacitly assumed that dislocations are spaced evenly along the boundary. If, for a given ϕ and a given small ψ, the necessary mean spacing is $15\frac{1}{4}$ lattice planes, we shall actually have groups of dislocations with spacings 15, 15, 15, 16, forming a ‘ superlattice ’. This small deviation will not alter the energy much, but if ψ is larger, so that the spacing should be, say, $3\frac{1}{2}$ planes, the actual spacings will be alternately 3 and 4. The energies given by (104)–(106) are calculated for those values of ψ and ϕ for which both x and y dislocations have integral spacings. Read and Shockley show that small deviations from this orientation cause the energy to rise steeply. The plot of energy against orientation shows a cusp similar to the $\psi \ln \psi$ cusp at the origin, but of smaller size. Since the integral spacings of dislocations of each type are destroyed by changes in ϕ as well as by changes in ψ, boundaries of this kind, in particular twin boundaries, have strongly preferred orientations in the two lattices.

5.3. LATTICES OF DISLOCATIONS

The idea of a lattice of dislocations was introduced by Taylor (1934), who considered that the distribution of positive and negative edge dislocations in a strained single crystal could be approximately represented by a regular lattice. Since the stresses used by Taylor were incorrect, the lattices he found to be stable differed somewhat from those we shall consider. The present treatment is in fact mathematically simpler, and the results are similar. However, our present ideas of the mechanism of work hardening suggest that arrays of dislocations which could reasonably be represented by such a lattice occur only in kink bands, and in these bands the dislocations are probably no longer free to move in their original glide planes. A model of more interest is that of a Frank–Read source in a finite crystal embedded in a matrix into which the dislocations cannot penetrate (§ 5.1).

Fig. 28



A lattice of edge dislocations. (The top dislocation at the left would more usually be represented by the symbol ⊥.)

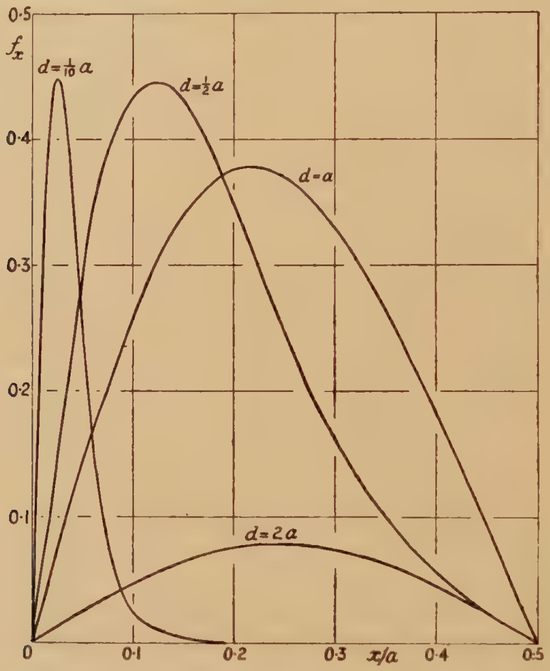
Figure 28 represents a lattice of positive and negative dislocations in which the positive dislocations lie on glide planes separated by d in columns at distances a. The negative dislocations lie in glide planes midway between these, in similar columns displaced a horizontal distance x. If the positive dislocations are held fixed, there will clearly be equilibrium when $x=0$ and when $x=\frac{1}{2}a$. It follows from eqns. (98) and (99) that there can be no equilibrium if all the dislocations are released, for the forces between any two columns are repulsive.

Possibly other arrays, in which positive and negative dislocations form close pairs, may be stable. When the positive dislocations in fig. 28 are fixed, the force on a negative dislocation is given by eqn. (99) as

$$\left.\begin{aligned} F_x &= \frac{\pi b D}{d} f_x \\ \text{where} \quad f_x &= \sum_{n=-\infty}^{\infty} \frac{\pi(x+na)}{d} \Bigg/ \cosh^2 \frac{\pi(x+na)}{d}. \end{aligned}\right\} \quad . \quad . \quad (107)$$

This force is plotted as a function of x/a for various values of d/a in fig. 29. The position $x=0$ is always one of unstable equilibrium, and $x=\frac{1}{2}a$ one of stable equilibrium. The maximum force $f_{\max}$, and the slope $\partial f/\partial x$ at $x=\frac{1}{2}a$ are given in the following table.

Fig. 29



The force f_x between the positive and negative dislocations of fig. 28, for various values of d/a.

d/a	0·1	0·5	1	2
$f_{\max}$	0·45	0·45	0·38	0·097
$-a\,\partial f/\partial x$	0·00	0·49	1·89	0·47
$(a/d)f^2_{\max}$	2·0	0·80	0·14	0·003

If a shear stress P is applied to the crystal, each negative dislocation is acted on by a force $-Pb$, and it comes to rest at a point x determined by the equation

$$F_x = Pb. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108)$$

If Pb is greater than $F_{\max}$, this equation has no root, and the lattice of negative dislocations moves through the crystal.

5.31. *Taylor's Theory of Work Hardening*

Taylor assumed that the elastic limit of a work-hardened single crystal was given by

$$Pb = F_{\max}$$

or

$$P = b\mu f_{\max}/2d(1-\sigma). \qquad (109)$$

He also assumed that at low temperatures the lattice of dislocations would be formed by dislocations moving into the crystal from mosaic walls separated by a distance L. The plastic strain during this movement is

$$\epsilon = Lb/ad. \qquad (110)$$

At higher temperatures the mosaic walls would be partly permeable to dislocations, so that some of the dislocations would have travelled distances greater than L before forming the array of fig. 28, and the plastic strain would be correspondingly greater.

Eliminating ad from eqns. (109) and (110), we obtain

$$\frac{P^2}{\epsilon} = \frac{b\mu^2}{4(1-\sigma)^2 L}\,\frac{a}{d}\,f^2{}_{\max}. \qquad (111)$$

The parabolic relation between stress and strain is in good general agreement with experimental results on cubic materials. The number of atoms in the side of a mosaic block is at low temperatures equal to L/b, or $\mu^2\epsilon/P^2$ multiplied by the numerical factor $af^2{}_{\max}/4(1-\sigma)^2 d$, which for moderate values of d/a lies between 0·1 and 1. For aluminium at $-185°$ C, the results of Boas and Schmid (1931) correspond to $\mu^2\epsilon/P^2 = 1{\cdot}56 \times 10^5$, giving $L/b \simeq 6 \times 10^4$, which is of the same order as the observed spacing of various imperfections in metals. The same agreement in order of magnitude is found for Al, Ca, Au, Fe and NaCl, at room temperature.

This agreement in form and order of magnitude is very satisfactory. However, the calculation considers the metal as having only one glide plane, whereas face-centred cubic metals have four glide planes. Single crystals of hexagonal metals have only one glide plane, and they show an increase of elastic limit with strain which is very much smaller than that calculated, and is approximately linear in the strain.

5.32. *Influence of Dislocations on the Shear Modulus*

Masing (1944, 1948) has pointed out that the presence of a lattice of dislocations reduces the shear modulus of the crystal. The displacement of the lattice of negative dislocations corresponding to eqn. (108) produces a reversible shear strain. If P is small, the displacement of the lattice is given by $\delta x\, dF/dx = Pb$, and the strain is $\epsilon' = b\delta x/ad$. This may be written

$$\epsilon' = 2(1-\sigma)P/(\mu a\, \partial f/\partial x),$$

while the elastic strain in a perfect crystal would be

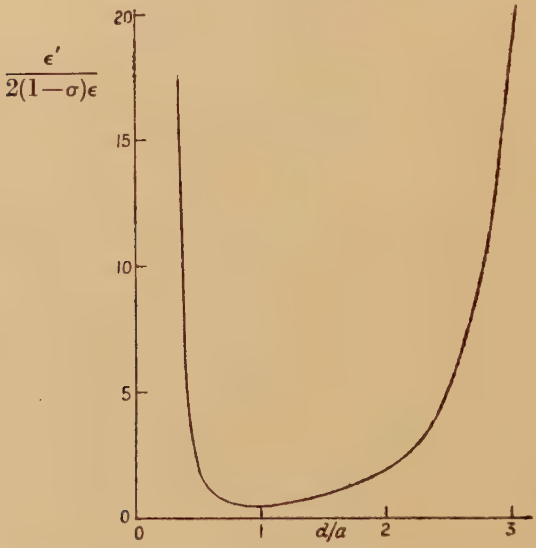
$$\epsilon = P/\mu.$$

The fractional change in modulus is

$$\epsilon'/\epsilon=2(1-\sigma)/(a\partial f/\partial x). \qquad (112)$$

This quantity is independent of the size of the mosaic blocks L, and of the density of dislocations in the lattice, which is proportional to $1/ad$. It depends only on the ratio d/a. Its value is plotted in fig. 30. Masing's results for large d/a are similar, but for small d/a he finds a value of ϵ'/ϵ which decreases steadily as d/a decreases. This is because he considers a lattice which would be stable only if the stress around a dislocation were that given by Taylor. This lattice, corresponding to $x=0$ in fig. 28, remains stiff if a increases with d held constant, whereas our lattice, corresponding to $x=\frac{1}{2}a$, becomes soft as a increases.

Fig. 30



Fractional change in modulus ϵ'/ϵ produced by the presence of the lattice of dislocations of fig. 28, for various values of d/a.

Equation (103) gives the restoring force on a screw dislocation escaping from a crossed grid of screws. Adding the repulsion $+b^2\mu/2\pi X$ of a dislocation occupying the missing place in the grid leads to a force between a crossed grid in the plane $x=0$ and a screw dislocation $(0, b, 0)$ passing through the point $(X, 0, 0)$ equal to

$$F_x=-\frac{b^2\mu}{2w}\left[1-\coth\frac{\pi X}{w}\right].$$

This is a repulsion diminishing exponentially with distance. On this basis a theory similar to that just given for a crystal containing sheets of edge dislocations may be developed for a crystal containing parallel crossed grids of screw dislocations, all right handed or all left handed.

Such a system forms an approximate model of a crystal which has been twisted about an axis perpendicular to the glide plane.

Since the maximum value of $a\partial f/\partial x$ in (112) is only about 2, the fractional change in modulus produced by a lattice of mobile dislocations is a large part of the modulus of a perfect crystal. In practice the moduli measured statically are changed only a few per cent by cold work. (They also agree closely with the moduli estimated from the speeds of ultrasonic waves, but this may not be significant because the frequencies of these waves are probably not high enough for the movement of dislocations to be suppressed.) We deduce that the system of dislocations in an ordinary metal is held in equilibrium by its interaction with immobile lattice defects and not merely by the forces between mobile dislocations. However, deviations of the order given by (112) have been observed by Crussard (1945) in zinc, by Crussard (1947) in magnesium, and by Andrade and Henderson (1951) in single crystals of gold and silver, using in all cases shears of the order of 0·001.

§6. THE INTERACTION OF DISLOCATIONS WITH FREE SURFACES

6.1. ELASTIC INTERACTION

If an elastic dislocation lies near to a free surface, it is less constrained than a dislocation in the interior of a specimen, and its energy is lower. There is thus a force attracting a dislocation to a free surface. This force was calculated for an edge dislocation in a circular cylinder by Koehler (1941); we shall treat in detail the simpler case of a screw dislocation, and then give an outline of Koehler's method. In both cases the force is approximately that due to an image dislocation of the same strength and opposite sign.

6.11. *The Screw Dislocation in a Circular Cylinder*

We use Eshelby's method of § 4.42. The sum of the displacements (57) and (59) is constant on any circle passing through the points $(\xi, 0)$ and $(R^2/\xi, 0)$. Therefore no stress acts across the orthogonal trajectories of these curves, which are the coaxal circles

$$[(x-\xi)^2+y^2]-\lambda[(x-R^2/\xi)^2+y^2]=0,$$

with these two points as limit points given by $\lambda=0$ and $\lambda=\infty$. The circle $x^2+y^2=R^2$ given by $\lambda=\xi^2/R^2$ is free from stress, and may again be taken as the outer boundary of the dislocated body. For the inner boundary required to form a dislocation we take $\lambda=\epsilon^2\xi^2/(R^4-\xi^4)$ where ϵ is small. This is a circle of radius ϵ around the point $(\xi, 0)$, with its centre displaced from that point by a distance of order ϵ^2. By considering the work done on the cut surfaces if the dislocation is made over a cut from $x=\xi+\epsilon$ to $x=R$, the energy is easily shown to be

$$W=\frac{b^2\mu}{4\pi}\ln\frac{R^2-\xi^2-\xi\epsilon}{R\epsilon}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

This is equal to $(b^2\mu/4\pi)\ln(R/\epsilon)$ when $\xi=0$ and the dislocation is at the centre of the cylinder, and is zero when $\xi=R-\epsilon$ and the inner and outer boundaries meet. The force on the dislocation is given by

$$F_x=-\frac{\partial W}{\partial \xi}=\frac{b^2\mu}{4\pi}\frac{2\xi+\epsilon}{R^2-\xi^2-\xi\epsilon}.$$

So long as the dislocation does not lie within a distance of order ϵ from either the centre or the boundary of the cylinder, this may be written

$$F_x=\frac{b^2\mu}{2\pi}\frac{\xi}{R^2-\xi^2}=b^2\mu/2\pi w,$$

where w is the distance between the dislocation and its image.

6.12. *The Edge Dislocation in a Circular Cylinder*

Koehler evaluated the energy in this case by working in bipolar coordinates. A small cylinder surrounding the line of the dislocation is again excluded from the range of integration, although in this case the surface of the small cylinder is not free from stress. The force is approximately

$$F_x=b^2\mu/2\pi(1-\sigma)w=bD/w,$$

where w is again the distance between the dislocation and its image.

6.2. INFLUENCE OF THE SURFACE ENERGY

6.21. *Energy of an Exposed Ledge*

When an edge dislocation enters a crystal from a free surface, it exposes a ledge of width b. If the surface energy is γ, the energy $b\gamma$ per unit length must be provided, in addition to the energy of the dislocation itself. An equal surface energy appears when the dislocation leaves the crystal. If the energy of the dislocation exceeds $b\gamma$ per unit length it is free to reach the surface ; if less, energy is required. For a complete dislocation in a close packed lattice b is the distance of a closest approach, equal to $a/\sqrt{2}$ in the cubic lattice and a in the hexagonal. If the dislocation is moving slowly and is a few atoms away from a free surface, its energy is about $\frac{1}{2}bD$ per unit length, where $D=b\mu/2\pi(1-\sigma)$.

This is compared with $b\gamma$ in the following table. The value of γ is that found experimentally for the liquid metal. Fisher and Dunn (1952) give the following 'fairly reliable' values for the surface energies of solid metals (units of dyne cm^{-1}): Ag 1 180, Cu 1 430, Au 1 550. These lie 30–50% above the values for the liquids.

Metal	10^{-8} cm	μ 10^{11} dyne cm^{-2}	σ	γ 10^2 dyne cm^{-1}	$\frac{1}{2}bD$ 10^{-5} erg cm^{-1}	$b\gamma$ 10^{-5} erg cm^{-1}
Ag	2·88	2·8	0·37	8·0	2·94	2·31
Au	2·88	2·8	0·42	5·8–10·0	3·18	1·70–2·88
Cd	2·97	2·1	0·30	6·3	2·11	1·88
Cu	2·55	4·4	0·34	11·0	3·45	2·81
Pb	3·49	0·6	0·44	4·4	1·04	1·29
Pt	2·77	6·3	0·39	18·2	6·30	5·04
Zn	2·66	3·6	0·23	7·4	2·63	1·98

Except in the case of Pb, the energy of the dislocation is always greater than that of the exposed surface. An isolated partial (§ 5.21 of Part III) has $\frac{1}{3}$ the energy of a complete dislocation, and exposes a ledge $1/\sqrt{3}$ times as wide. The value of $b\gamma$ now exceeds that of $\frac{1}{2}bD$ in all cases. Such a crude calculation does not justify quantitative conclusions, but it is clear that a dislocation may require energy in order to break through the surface of even a perfect crystal.

A screw dislocation does not expose a complete ledge as it enters or leaves the specimen, but gradually exposes a ledge as it moves. There is therefore no energy required to form or destroy it, but each end is acted on by a force $b\gamma$ tending to shorten the exposed ledges.

6.22. *Hollow Dislocations and Capillary Equilibrium*

Frank (1951 b) has examined the condition that a dislocation should have less energy when its core is hollow than when it is filled with strained material. If a screw dislocation of Burgers vector b lies along the axis of a cylinder of radius R, the elastic energy of unit length is of the form

$$E_1=(b^2\mu/4\pi)\ln(R/\theta b). \qquad (114)$$

The value of θ, which is of order unity, depends on the law of force in the core of the dislocation; Frank considers various laws. If the dislocation is hollow, with radius r so large that Hooke's law applies accurately beyond this radius, there is an elastic energy $(b^2\mu/4\pi)\ln(R/r)$, and a surface energy $2\pi\gamma r$, where γ is the surface energy, giving a total energy

$$E_2=(b^2\mu/4\pi)\ln(R/r)+2\pi\gamma r. \qquad (115)$$

As r varies, E_2 passes through a minimum for

$$r=r_0=b^2\mu/8\pi^2\gamma. \qquad (116)$$

The quantity γ/μ, which is a measure of the range of intermolecular forces, varies rather little from one substance to another, and is about $\frac{1}{4}$ Å. Thus $r_0>b$ only if $b>20$ Å, and unless this condition is satisfied the dislocation will certainly not be hollow. If it is satisfied, the dislocation will be hollow only if the minimum value of E_2 given by (115) and (116) is less than E_1, which requires that $b>8\pi^2\theta\gamma/\mu$. Since usually $\theta<1$, this condition is satisfied if $r_0>b$. We conclude that dislocations will be hollow if b exceeds about 20 Å (Frank obtains 10 Å as a more reliable estimate), but not otherwise. Verma (1951) has found hollow dislocations in carborundum ($b=15$ Å). In a long chain paraffin ($b=47$ Å) Dawson and Vand (1951) found no hollows. Frank has pointed out that in the latter case the energy of the core of the dislocation is very low, because two long chain molecules have many positions with energy only slightly higher than that of their proper positions in the crystal. This means that θ in (114) is large, and the condition $b>8\pi^2\theta\gamma/\mu$ is not satisfied.

Frank also considered the way in which a hollow dislocation widens out when it meets a free surface, and the shape of the depression in the surface where an ordinary solid-cored dislocation emerges. If the surface energy is independent of orientation, the depression is shallow in form, but of infinite volume and depth. If the surface is a natural habit face of the crystal, so that a slight change in the orientation of the surface exposed makes a first order change in the energy of the surface, the surface depression may be completely eliminated if b lies in the surface, and is small in almost all other cases.

§7. The Line Energy of Dislocations

The energy of an elastic dislocation of arbitrary shape may be calculated by the methods of Volterra and Burgers. We shall illustrate this in § 7.1 by considering an infinite body containing a dislocation of which the singular line is a circle of radius R. The body is made doubly connected by a toroidal hole of radius $\epsilon \ll R$ enclosing the singular line, and the Burgers vector $\mathbf{b}$ of the dislocation lies in the plane of the circle. This corresponds to Orowan's dislocation of fig. 4. In this treatment, the energy increases indefinitely as ϵ is diminished, and it would be more satisfactory to have a treatment based on an approximation such as that of Peierls. As will be seen from §§ 2 and 4 of Part III, these closer approximations have been applied only to a straight dislocation and to a dislocation which deviates only slightly from a straight line.

Most theories of plastic deformation are based on models in which a dislocation is formed and travels through the lattice under the influence of an applied stress until it meets some type of obstacle. Either in the process of formation or in overcoming this obstacle, the dislocation can act only with the help of thermal vibrations. Different models give different dependences of the necessary activation energy on the applied stress, and an analysis of creep tests may indicate which model is appropriate to a particular type of deformation. In §7.2 we calculate the activation energy for a number of models. A related problem which is still unsolved is the calculation of the appropriate mean stress which opposes the motion of a dislocation through a lattice (such as that of a metallic solid solution) containing stresses which fluctuate over very short distances. Two approaches, leading to different results, are discussed in § 7.3.

7.1. THE ELASTIC ENERGY OF A CIRCULAR DISLOCATION

We take the origin of Cartesian coordinates at the centre of the dislocation of § 7, with the z axis normal to the plane of the circle, and the x axis parallel to the slip direction. Then the Burgers vector is $\mathbf{b}=(b, 0, 0)$. Let the polar angle θ of a point $(\xi, \eta, 0)$ on the circle be given by $\tan\theta=\eta/\xi$. Then the displacement $\mathbf{u}$ at any point is given by (35) of § 3.4.

We have

$$\mathbf{u}^{*}=(\Omega/4\pi)(b, 0, 0)$$

$$\mathbf{u}^{**}=\int_0^{2\pi}(0, 0, b\cos\theta)R\,d\theta/4\pi r$$

and
$$\psi=-\frac{\lambda+\mu}{4\pi(\lambda+2\mu)}\int_0^{2\pi} bz\cos\theta\, R\,d\theta/r,$$

where
$$r^2=(x-R\cos\theta)^2+(y-R\sin\theta)^2+z^2.$$

The elastic energy of the dislocation is

$$W = -\tfrac{1}{2} b \iint p_{xz}\, dx\, dy, \qquad \ldots\ldots \qquad (117)$$

where the integral is taken over the circle $x^2+y^2 \leqslant (R-\epsilon)^2$, $z=0$. Since $p_{xz} = \mu(\partial u_z/\partial x + \partial u_x/\partial z)$, the energy may be written

$$W = -\frac{1}{8\pi} b^2 \mu \iint \frac{\partial \Omega}{\partial z}\, dx\, dy - \frac{1}{8\pi} R b^2 \mu \iiint \frac{\partial}{\partial x}\left(\frac{\cos\theta}{r}\right) d\theta\, dx\, dy$$

$$+ \frac{\lambda+\mu}{4\pi(\lambda+2\mu)} R b^2 \mu \iiint \frac{\partial^2}{\partial x \partial z}\left(\frac{z\cos\theta}{r}\right) d\theta\, dx\, dy. \qquad \ldots \qquad (118)$$

We consider first the integral involving Ω, which is equal to the scalar magnetic potential of a unit current flowing in the circle of radius R. It is easily seen that the integral is the mutual inductance of the circles of radii R and $R-\epsilon$, taken with negative sign, and this mutual inductance may be evaluated as the line integral of the vector potential of the larger circle, taken round the smaller circle. This is

$$-2\pi(R-\epsilon) \int_0^{2\pi} \frac{\cos\theta\, R d\theta}{[R^2+(R-\epsilon)^2-2R(R-\epsilon)\cos\theta]^{1/2}}.$$

To evaluate this when $\epsilon \ll R$, put $\sin \tfrac{1}{2}\theta = t$, when the integral becomes

$$-8\pi R^2 \int_0^1 \frac{(1-2t^2)\, dt}{(1-t^2)^{1/2}(\epsilon^2+4R^2t^2)^{1/2}}.$$

In the region $0 \leqslant t \leqslant (\epsilon/2R)^{1/2}$ the integrand is approximately $(\epsilon^2+4R^2t^2)^{-1/2}$, while in the region $(\epsilon/2R)^{1/2} < t \leqslant 1$ the integrand is approximately $(1-2t^2)/2Rt(1-t^2)^{1/2}$.

Evaluating these two separately, we obtain a contribution to the energy of

$$W_1 = Rb^2\mu(\tfrac{1}{2}\ln R/\epsilon + \ln 2 - 1).$$

The integrands of the second and third terms in W reduce in the plane $z=0$ to $-(x-R\cos\theta)\cos\theta/[(x-R\cos\theta)^2+(y-R\sin\theta)^2]^{3/2}$.

Integration with respect to x gives $\cos\theta/[(x-R\cos\theta)^2+(y-R\sin\theta)^2]^{1/2}$, taken between the limits $x = \pm[(R-\epsilon)^2-y^2]^{1/2}$. Write $x=(R-\epsilon)\cos\alpha$, $y=(R-\epsilon)\sin\alpha$, and the integrals become

$$(R-\epsilon)\int_{\alpha=-\pi/2}^{\pi/2}\int_{\theta=0}^{2\pi}\left[\frac{1}{\{R^2+(R-\epsilon)^2-2R(R-\epsilon)\cos(\theta-\alpha)\}^{1/2}}\right.$$

$$\left. - \frac{1}{\{R^2+(R-\epsilon)^2+2R(R-\epsilon)\cos(\theta+\alpha)\}^{1/2}}\right]\cos\theta\cos\alpha\, d\theta\, d\alpha$$

$$= \pi(R-\epsilon)\int_0^{2\pi}\frac{\cos\phi\, d\phi}{[R^2+(R-\epsilon)^2-2R(R-\epsilon)\cos\phi]^{1/2}}.$$

The contribution of these terms to the energy is thus $[(\lambda+\mu)/(\lambda+2\mu)-\tfrac{1}{2}]$ times the contribution of the first term, giving a total energy of

$$W = 2\pi R \frac{b^2\mu}{4\pi}\left[\tfrac{1}{2} + \frac{\lambda+\mu}{\lambda+2\mu}\right](\ln R/\epsilon + 2\ln 2/e). \qquad \ldots \qquad (119)$$

The factor in square brackets is $\frac{1}{2}[1+1/(1-\sigma)]$, the mean of the corresponding factors for a straight screw dislocation and a straight edge dislocation. To estimate the energy of a circular dislocation in a cubic crystal lattice we set $\epsilon=2\zeta/e$, where $\zeta=b/2(1-\sigma)$, by analogy with the value found in § 2.2 of Part III for a pair of straight dislocations.

7.2. ACTIVATION ENERGY PROBLEMS

In most models of the creep process, a dislocation will travel without any activation energy if the applied stress reaches some value P_1, *the critical stress*. Under a lower *applied stress* P_0 it will move only with an *activation energy* $U(P_0, P_1)$. Since $U=0$ when $P_0=P_1$, it is convenient to express U as a function of P_1 and the *activation stress* P_1-P_0. If we make the reasonable assumption that in similar tests at different temperatures T the creep rate always becomes noticeable when

$$U(P_1-P_0, P_1)=mkT, \qquad \ldots\ldots \quad (120)$$

where m is a constant independent of P_0, P_1 and T, and k is Boltzmann's constant, we see that creep experiments at different temperatures give some information on the form of the function U.

Wyatt (1951) has shown that if creep occurs by a pure exhaustion process (Mott and Nabarro 1948, Smith 1948), only dislocations with a narrow range of values of P_1 contribute to the creep rate at any given time t after the start of the test, and that the activation energy U of the active group obeys the law

$$\frac{dU}{dt}=\frac{kT}{t}. \qquad \ldots\ldots\ldots\ldots \quad (121)$$

Suppose the specimen is allowed to creep under a load $P_{0\alpha}$ up to a time t_α, and the load is then slightly increased to $P_{0\beta}$. The activation stress and activation energy for the group of dislocations active at this time are decreased, and the creep rate increases to the larger value which it had at some earlier time t_β. We now assume that U depends very little on P_1 in the range considered, and that the total number of dislocations available in a range from P_1 to P_1+dP_1 and also the total amount of slip caused by each dislocation when it moves are almost independent of P_1. It follows that the activation energies, and hence the activation stresses, are the same when a load P_β is applied after the specimen has crept for a time t_α under a load P_α and when a constant load P_α has been applied for a time t_β.

Thus
$$P_1(t_\alpha)-P_{0\beta}=P_1(t_\beta)-P_{0\alpha}$$
or
$$P_1(t_\alpha)-P_1(t_\beta)=P_{0\beta}-P_{0\alpha}, \qquad \ldots\ldots\ldots \quad (122)$$

a known quantity. It also follows from (121) that

$$U(t_\alpha)-U(t_\beta)=kT(\ln t_\alpha-\ln t_\beta). \qquad \ldots\ldots \quad (123)$$

Comparison of (122) and (123) gives the form of the function $U(P_1 - P_0)$. The difficulty in applying the method in practice is that the experimentally available range of P_0 is so small that the function is almost linear in this range.

We now calculate the function $U(P_1 - P_0, P_0)$ for a number of models. In § 7.21 we take account of the fact that the line energy of a dislocation is not strictly constant, but in other cases we assume a constant energy E for unit length.

7.21. *The Models of Becker and Orowan*

The original model of Becker (1925) and Orowan (1934 a) does not provide a detailed mechanism for the nucleation of slip, but assumes that slip will begin if the shear stress in a volume V reaches a critical value P_1. In the absence of an external stress this requires an energy $VP_1^2/2\mu$, where μ is the shear modulus. In the presence of an external stress P_0, this is reduced to

$$U = V(P_1 - P_0)^2/2\mu. \qquad \qquad (124)$$

If observable slip begins when $U = mkT$, the strength at temperature T is given by

$$P_0 = P_1 - (2\mu mk/V)^{1/2}T^{1/2}. \qquad \qquad (125)$$

We may also consider the specific mechanism proposed by Orowan (1934 b), in which the nucleation of slip occurs by the formation of a loop of dislocation in a region of the crystal previously perfect. We assume that the region in which the loop is formed is determined by the presence of a residual stress P^* in the same direction as the applied stress P_0, and also of a stress concentration factor q. The total stress on the loop is $P = P^* + qP_0$. When the radius of the loop is R, the total energy of the specimen is $W - \pi R^2 b(P^* + qP_0)$, where W is given by (119).

In the next approximation, the circular dislocation cannot be in equilibrium under the stress P, for this stress exerts an equal outward force bP on each element of its length. Since the energies per unit length of the edge and screw elements are in the ratio $1 : 1 - \sigma$, their radii of curvature must be in the same ratio. The equilibrium shape of the dislocation is roughly elliptical, with the minor axis in the glide direction, and shorter than the major axis by a factor $(1-\sigma)^{1/3}$. If we assume the circular form, the total energy of the specimen increases by $W - \pi R^2 bP$, which is positive for small R, reaches a maximum, and becomes negative for large R. The radius R_m of the circle of maximum energy, and its energy U, are given by

$$\frac{P}{\mu} = \frac{1}{8\pi}\left(1 + \frac{1}{1-\sigma}\right)\frac{b}{R_m}\ln\frac{2R_m}{\zeta} \qquad \qquad (126)$$

and

$$\frac{U}{b^3\mu} = \frac{1}{8}\left(1 + \frac{1}{1-\sigma}\right)\frac{R_m}{b}\left(\ln\frac{2R_m}{\zeta} - 2\right). \qquad \qquad (127)$$

Numerical values have been obtained by Frank (1950 a). He replaced our factor of $1 + 1/(1-\sigma)$ by 2, and considered the values $\zeta = 2b/e$ and $\zeta = 4b/e$ where we take $\zeta = b/2(1-\sigma)$. His results are shown in fig. 31.

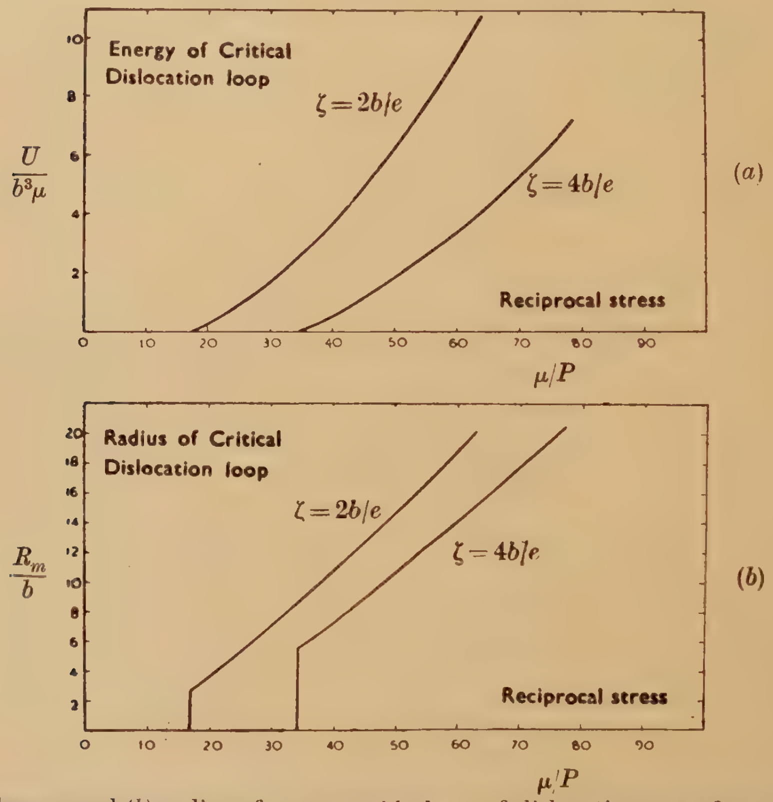
Since $b^3\mu$ is of order 5 ev, and thermally activated processes occur very rarely at room temperature if U is greater than about $1\frac{1}{4}$ ev, loops will be formed spontaneously only if P/μ is close to the critical value, which is about 1/20 or 1/30. For a soft metal such as sodium, where $b^3\mu$ is less than 1 ev, loops may be formed when P/μ is about 1/80.

7.22. *The Model of Mott and Nabarro*

Mott and Nabarro (1948) considered a straight element of dislocation of length L moving in a periodic stress field, so that the total force on the dislocation when it has moved a distance x is

$$P_1bL\cos(2\pi x/\Lambda). \qquad (128)$$

Fig. 31



(*a*) Energy and (*b*) radius of a metastable loop of dislocation as a function of the applied stress, for two values of the energy in the core of a dislocation (after Frank).

An applied stress P_0 exerts a force P_0bL on the element, which thus has stable and unstable positions of equilibrium given by alternate roots of the equation

$$P_1\cos(2\pi x/\Lambda)=P_0. \qquad (129)$$

If P_0 is nearly equal to P_1, a pair of roots is given by

$$2\pi x/\Lambda=\pm[2(P_1-P_0)/P_1]^{1/2}. \qquad (130)$$

The activation energy is

$$U=\frac{2\sqrt{2}}{3\pi}bL\Lambda\frac{(P_1-P_0)^{3/2}}{P_1^{1/2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

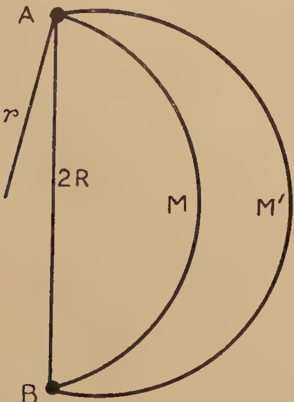
and this is equal to mkT when

$$P_0=P_1-(3\pi mkP^{1/2}/2\sqrt{2}bL\Lambda)T^{2/3}. \quad . \quad . \quad . \quad . \quad (132)$$

7.23. *The Models of Orowan, and of Frank and Read*

Orowan (1948) proposed a model of an over-aged precipitation-hardened alloy in which the nucleation of slip occurred by the bursting of a loop of dislocation held fixed by particles of precipitate at two points a distance $2R$ apart. Frank and Read (1950) introduced a model which would give an unlimited amount of slip from a single source.

Fig. 32



A stable circular arc of dislocation AMB and an unstable arc AM′B, each of radius r and closing a gap AB of width $2R$.

The mechanism by which a nucleus of slip starts to grow is the same as that suggested by Orowan. This type of nucleus has the remarkable property that its behaviour is determined solely by the one parameter R.

Assuming that the line energy E is constant, the critical stress P_1 is given by

$$P_1=E/bR. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (133)$$

For this stress, the loop of dislocation in equilibrium forms a semicircle of radius R. At a smaller applied stress P_0 there are two positions of equilibrium (fig. 32), both circular arcs of radius r given by

$$r=E/bP_0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

The shorter, stable arc subtends at its centre an angle $2\sin^{-1}(R/r)$. Its line energy is $2Er\sin^{-1}(R/r)$. As it has moved from its original position as a straight line joining the two points to its present position, the applied stress P_0 has done work $bP_0[r^2\sin^{-1}(R/r)-R(r^2-R^2)^{1/2}]$.

The total energy of the stable arc is thus

$$2Er\sin^{-1}(R/r)-bP_0[r^2\sin^{-1}(R/r)-R(r^2-R^2)^{1/2}].$$

By means of (133) and (134) this may be reduced to the form

$$\frac{E^2}{bP_0}\left[\sin^{-1}\frac{P_0}{P_1}+\frac{P_0(P_1^2-P_0^2)^{1/2}}{P_1^2}\right]. \quad . \quad . \quad . \quad . \quad . \quad (135)$$

The line energy of the larger, unstable, arc is $2Er[\pi-\sin^{-1}(R/r)]$, while the work done in its formation is

$$bP_0[r^2\{\pi-\sin^{-1}(R/r)\}+R(r^2-R^2)^{1/2}].$$

The total energy reduces to

$$\frac{E^2}{bP_0}\left[\pi-\sin^{-1}\frac{P_0}{P_1}-\frac{P_0(P_1^2-P_0^2)^{1/2}}{P_1^2}\right]. \quad . \quad . \quad . \quad . \quad (136)$$

The activation energy U is the difference of (136) and (135), given by

$$U=\frac{2E^2}{bP_0}\left[\cos^{-1}\frac{P_0}{P_1}-\frac{P_0(P_1^2-P_0^2)^{1/2}}{P_1^2}\right]. \quad . \quad . \quad (137)$$

If P_0 is very nearly equal to P_1, this becomes

$$U=\frac{8\sqrt{2}E^2}{3b}\frac{(P_1-P_0)^{3/2}}{P_1^{5/2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

This model, like that of Mott and Nabarro, leads to $U\propto(P_1-P_0)^{3/2}$. The essential reason for this is that Mott and Nabarro take the length of dislocation which is moving to be a constant L, while in the present model the length again tends to a finite value πR as P_0 tends to P_1.

If slip is observed when $U=mkT$, we have

$$P_0=P_1-(3bmk/8\sqrt{2}E^2)^{2/3}P_1^{5/3}T^{2/3}. \quad . \quad . \quad . \quad (139)$$

Thermal agitation allows slip to be observed when P_0 is less than P_1 by a fraction

$$\frac{P_1-P_0}{P_1}=\left(\frac{3bP_1mkT}{8\sqrt{2}E^2}\right)^{2/3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (140)$$

Taking as a lower limit for E the value $0{\cdot}16\,b^3\mu$ and assuming $m=40$ leads to

$$\frac{P_1-P_0}{P_1}=56\left(\frac{P_1}{\mu}\cdot\frac{kT}{b^3\mu}\right)^{2/3}.$$

If we take $P_1=1$ kg mm^{-2}, $\mu=4\times10^{11}$ dyne cm^{-2}, $b=3\times10^{-8}$ cm, $T=300°$ K, this becomes

$$(P_1-P_0)/P_1=5{\cdot}5\times10^{-3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

Thus, on this model, the strength should be very insensitive to temperature, and the assumption that P_1-P_0 is small is always justified.

7.24. *The Model of Cottrell and Bilby*

In the model of Cottrell and Bilby, a long straight dislocation approaches a region of adverse stress, similar to that considered by Mott and Nabarro. In the application which they considered, a dislocation is bound to a row of solute atoms by a force which for small displacements increases linearly with the displacement, and with increasing displacement reaches a maximum and decreases to zero at large distances. We consider only the behaviour of the dislocation when it lies near to the region of maximum restoring force. The nucleation of slip occurs by a bulge of dislocation spilling over the line of maximum stress by thermal activation. There is a form of bulge which is in equilibrium at all points, the equilibrium at some points being unstable. Once this is formed, further slip can occur without activation. The width of the critical bulge depends on the activation stress in a manner to be determined.

The analysis is tractable only if we assume that the critical bulge always has a small slope. That is to say, if x is the coordinate in the glide plane along which the internal stress is constant, and y is the direction of motion of the dislocation,

$$dy/dx \ll 1. \qquad (142)$$

The curvature of the bulge at any point is then d^2y/dx^2. We take the maximum value of the internal stress to be P_1, while the stress at points near the maximum is $P_1 - \kappa^2 y^2$. If an external stress P_0 is applied, the form of a dislocation which is in equilibrium at all points is given by

$$E d^2y/dx^2 = b(P_1 - P_0 - \kappa^2 y^2). \qquad (143)$$

Two solutions of this are obvious. The first,

$$y = -y_0 = -(P_1 - P_0)^{1/2}/\kappa, \qquad (144)$$

is a straight dislocation in stable equilibrium at all points, which has not crossed the region of maximum stress. The second solution, $y = +y_0$, represents a straight dislocation entirely in unstable equilibrium. The critical form coincides with (144) at points far from the centre of the bulge, and has $y > 0$ near the middle of the bulge. To integrate (143) we multiply both sides by 6 dy/dx, and fix the constant of integration by the condition that $dy/dx = 0$ if $y = -y_0$. The result is

$$3E(dy/dx)^2 = 6b(P_1 - P_0)(y + y_0) - 2b\kappa^2(y^3 + y_0^3). \qquad (145)$$

In general, the solution of an equation of this form is a Weierstrassian elliptic function. In this particular case the solution required is elementary,

$$y = y_0(3\ \mathrm{sech}^2\, x/x_0 - 1), \qquad (146)$$

where

$$x_0 = [4E^2/b^2\kappa^2(P_1 - P_0)]^{1/4}, \qquad (147)$$

as may be verified by substitution. The maximum value of dy/dx is $4y_0/\sqrt{3}x_0$, which is proportional to $(P_1 - P_0)^{3/4}$. As P_0 tend to P_1, this tends to zero, and (142) is thus satisfied.

The width of the bulge, x_0, tends to infinity as the activation stress $(P_1 - P_0)$ tends to zero. We therefore expect that the activation energy

U will tend to zero with (P_1-P_0) more slowly than in the models of §§ 7.22 and 7.23. The line energy of a bulged dislocation exceeds that of a straight dislocation by

$$\tfrac{1}{2}E\int_{-\infty}^{\infty}(dy/dx)^2\,dx.$$

For the critical bulge (146) this is

$$(18Ey_0^2/x_0^2)\int_{-\infty}^{\infty}\operatorname{sech}^4(x/x_0)\tanh^2(x/x_0)\,dx=24Ey_0^2/5x_0. \quad . \quad (148)$$

The work done by all the stresses, internal and external, on an element of the dislocation as it moves forward is

$$b\int_{-y_0}^{y}(P_0-P_1+\kappa^2\eta^2)\,d\eta=b[(P_0-P_1)(y+y_0)+\tfrac{1}{3}\kappa^2(y^3+y_0^3)].$$

Integrating over the length of the dislocation, we find the total work for the critical bulge to be

$$-9b\kappa^2y_0^3\int_{-\infty}^{\infty}\operatorname{sech}^4(x/x_0)\tanh^2(x/x_0)\,dx$$

$$=-12b\kappa^2x_0y_0^3/5=-24Ey_0^2/5x_0. \quad . \quad . \quad . \quad . \quad (149)$$

The activation energy is the difference of (148) and (149).

$$U=\frac{48Ey_0^2}{5x_0}=\frac{24}{5}\left(\frac{2Eb}{\kappa^3}\right)^{1/2}(P_1-P_0)^{5/4}. \quad . \quad . \quad . \quad (150)$$

If slip becomes observable when $U=mkT$, the strength is given as a function of temperature by

$$P_0=P_1-(25m^2k^2\kappa^3/1152Eb)^{2/5}T^{4/5}. \quad . \quad . \quad . \quad . \quad (151)$$

In the case considered by Cottrell and Bilby, the stress is given in terms of the displacement y' from the position of tightest binding by

$$P=2A\rho y'/(y'^2+\rho^2)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

where ρ is a length estimated to be 2×10^{-8} cm. The maximum stress P_1 occurs when $y'=\rho/\sqrt{3}$, and is

$$P_1=3\sqrt{3}A/8\rho^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (153)$$

At this point dP/dy' vanishes, and $d^2P/dy'^2=-9P_1/2\rho^2$. If we approximate to this law by a parabolic law giving the same maximum stress and the same value of d^2P/dy^2 at the point of maximum stress, we find

$$\kappa^2=9P_1/4\rho^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (154)$$

The best estimate of P_1 is (Cottrell 1952 a) $P_1=6\times10^{10}$ dyne cm^{-2}. Taking $\mu=8{\cdot}28\times10^{11}$ dyne cm^{-2}, $b=2{\cdot}48\times10^{-8}$ cm, we estimate $E=0{\cdot}16b^2\mu=8{\cdot}13\times10^{-5}$ dyne. If we assume $m=50$, eqn. (151) becomes

$$P_0=6\times10^{10}-4\times10^8\,T^{4/5}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

where P_0 is the resolved shear stress in dyne cm^{-2} (half the yield strength for a crystal with a glide system favourably oriented), and T is the absolute temperature in degrees Kelvin. At room temperature, $T=300°$ K, and $P_0=2{\cdot}2\times10^{10}$ dyne cm^{-2}. At 80° K, $P_0=4{\cdot}7\times10^{10}$ dyne cm^{-2}, which is already 0·78 of its limiting value at low temperatures. The present approximation, which assumes that $dy/dx\ll1$, is always valid when $P_0\rightarrow P_1$. The greatest value of dy/dx, namely $4y_0/\sqrt{3}x_0$, occurs when $P_0=0$, and is $\frac{4}{3}\,(b\rho P_1/E)^{1/2}=0{\cdot}8$. Even in this limiting case the error is not likely to be very serious, since the greatest value of dy/dx is multiplied by the vanishing factor d^2y/dx^2.

7.25 *A Model of an Over-aged Alloy*

Orowan's model of an over-aged alloy considers a dislocation held rigidly at two points, and a bulge forming and bursting between the points. This model should apply to an over-aged alloy in which the particles of precipitate have become incoherent with the lattice of the matrix, and there are no longer any internal strains. Where the misfit between the lattices of the matrix and the precipitate is small, over-ageing may occur by particles of precipitate growing to a large size, while still remaining coherent. Dislocations moving in the resulting stress field will move most freely in the regions of the lattice furthest from particles of precipitate, but they are not held at fixed points close to the parts which move. We are thus led to consider a model in which a dislocation moving in the y direction is held up near the line $y=0$ by an adverse internal stress: this adverse stress is somewhat lower in one region of this line (which we take to lie near $x=0$) than elsewhere. The exact form of the field of internal stress cannot greatly affect the general physical behaviour, and our choice is governed by mathematical convenience. The way in which a suitable function was found is indicated at the end of this section. We actually take the internal stress to be

$$P_1-\kappa^2y^2+(4E^2/\kappa^2b^2x_1^4)[1-\tfrac{1}{4}(9-\gamma^2)\,\mathrm{sech}^4\,(x/x_1)], \quad . \quad . \quad (156)$$

where γ and x_1 are arbitrary constants.

Assuming as before that $dy/dx\ll1$, the shape of the dislocation is given by

$$Ed^2y/dx^2=b\{P_1-P_0-\kappa^2y^2+(4E^2/\kappa^2b^2x_1^4)[1-\tfrac{1}{4}(9-\gamma^2)\,\mathrm{sech}^4\,(x/x_1)]\}. \quad (157)$$

We now look for approximate solutions of this equation which will be exact in the limit $P_0\rightarrow P_1$. The solution corresponding to (144) is

$$y_1=(2E/\kappa^2bx_1^2)[\tfrac{1}{2}(3-\gamma)\,\mathrm{sech}^2\,(x/x_1)-1], \quad . \quad . \quad . \quad . \quad (158)$$

while the solution corresponding to (146) is

$$y_2=(2E/\kappa^2bx_1^2)[\tfrac{1}{2}(3+\gamma)\,\mathrm{sech}^2\,(x/x_1)-1]. \quad . \quad . \quad . \quad . \quad (159)$$

If $\gamma=3$, the dip in the potential ridge vanishes, and (158) and (159) reduce to (144) and (146). In general, the critical stress is not equal to P_1, and there is a positive activation energy when $P_0=P_1$. There is no simple way of determining the actual critical stress, or of finding the activation energy when the applied stress is just less than the critical stress. In the particular case $\gamma=0$, the solutions (158) and (159) coincide, which means that the critical stress in this case actually is P_1. For $P_0=P_1$, the equation

$$Ed^2y/dx^2=b\{P_1-P_0-\kappa^2y^2+(4E^2/\kappa^2b^2x_1^4)[1-\tfrac{9}{4}\operatorname{sech}^4(x/x_1)]\} \qquad (160)$$

has two coincident solutions

$$y^*=(2E/\kappa^2bx_1^2)[\tfrac{3}{2}\operatorname{sech}^2(x/x_1)-1]. \qquad (161)$$

If the applied stress P_0 is now taken slightly less than P_1, the solutions y^* separate into two solutions again representing the stable and the unstable equilibrium forms of the dislocation. These solutions may be written

$$y=y^*+\eta, \qquad (162)$$

where η is small. We substitute (162) into (160) and find that η satisfies the equation

$$E\eta''+2\kappa^2by^*\eta=b(P_1-P_0-\kappa^2\eta^2), \qquad (163)$$

or, if we write

$$x/x_1=\xi, \qquad (164)$$

$$d^2\eta/d\xi^2+(6\operatorname{sech}^2\xi-4)\eta=bx_1^2(P_1-P_0-\kappa^2\eta^2)/E. \qquad (165)$$

Since the solutions y^* coincide, they represent states of neutral equilibrium. We therefore expect that the dominant term in η will be of order $(P_1-P_0)^{1/2}$. This means that the terms on the left of (165) are of larger order than those on the right, and must vanish to order $(P_1-P_0)^{1/2}$. Thus the dominant term in η is a solution of the homogeneous equation $\eta''+(6\operatorname{sech}^2\xi-4)\eta=0$. This equation has the bounded, even, solution $\eta=\operatorname{sech}^2\xi$, and an unbounded, odd, solution $\operatorname{sech}^2\xi\int^{\xi}\cosh^4u\,du$, which we reject. We thus try for η a solution of the form

$$\eta=C(P_1-P_0)^{1/2}\operatorname{sech}^2\xi+(P_1-P_0)\zeta(\xi), \qquad (166)$$

where the constant C and the function ζ remain finite as $P_0\to P_1$. To order (P_1-P_0), eqn. (165) reduces to

$$\zeta''+(6\operatorname{sech}^2\xi-4)\zeta=(bx_1^2/E)(1-\kappa^2C^2\operatorname{sech}^4\xi). \qquad (167)$$

Again the homogeneous equation has a solution $\zeta=\operatorname{sech}^2\xi$, and we look for a general solution of the form

$$\zeta=u\operatorname{sech}^2\xi. \qquad (168)$$

Then u must satisfy

$$u''\operatorname{sech}^2\xi-4u'\operatorname{sech}^2\xi\tanh\xi=(bx_1^2/E)(1-\kappa^2C^2\operatorname{sech}^4\xi).$$

This equation has the integrating factor $\operatorname{sech}^2\xi$, leading to the integral

$$u'=(bx_1^2/E)\cosh^4\xi\int_{\xi_0}^{\xi}(1-\kappa^2C^2\operatorname{sech}^4\chi)\operatorname{sech}^2\chi\,d\chi. \qquad (169)$$

We are looking for solutions in which ζ, and therefore u, is an even function of ξ. It follows that we must take $\xi_0=0$. Moreover, for large $|\xi|$, ζ must be bounded, u must be $O\,[\exp 2|\xi|]$, and u' of the same order.

The integral in (169) is elementary, and it is easily seen that $u' \sim \exp 4|\xi|$ unless $\kappa^2C^2=15/8$. This choice leads to

$$u'=-(bx_1^2/8E)(4 \sinh \xi \cosh \xi+3 \tanh \xi).$$

Integrating again, we find

$$\left.\begin{aligned} u&=-(bx_1^2/8E)(2 \cosh^2 \xi+3 \ln \cosh \xi)+u_0 \\ \zeta&=-(bx_1^2/8E)(2+3 \operatorname{sech}^2 \xi \ln \cosh \xi)+u_0 \operatorname{sech}^2 \xi. \end{aligned}\right\} \quad . \quad . \quad (170)$$

Since we have postulated that ζ remains finite as $P_0 \to P_1$, the last term in (170) is of the same form as the dominant term in η, but of smaller order. We therefore neglect it, and obtain the pair of solutions

$$\begin{aligned} \eta=&\pm[15(P_1-P_0)/8\kappa^2]^{1/2} \operatorname{sech}^2 \xi \\ &-[bx_1^2(P_1-P_0)/8E](2+3 \operatorname{sech}^2 \xi \ln \cosh \xi). \end{aligned} \quad . \quad . \quad . \quad (171)$$

The two solutions differ by a bulge of width about $2x_1$, which moves forward a distance $[15(P_1-P_0)/2\kappa^2]^{1/2}$. This behaviour is of the same form as in the models of §§ 7.22 and 7.23, and the activation energy is again proportional to $(P_1-P_0)^{3/2}$.

We now indicate the argument by which a suitable form of stress field was found. In the limit $P_0=P_1$, the shape of the dislocation is governed by an equation of the form

$$Y''+2Y^2=f(X) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (172)$$

where X and Y are constant multiples of x and y respectively. We wish to know two solutions of this equation. Let them be Y_1 and $Y_1+\Delta$. Then

$$Y_1''+2Y_1^2=f(X) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (173)$$

and

$$Y_1''+\Delta''+2Y_1^2+4Y_1\Delta+2\Delta^2=f(X). \quad . \quad . \quad . \quad (174)$$

Subtracting (173) from (174), we find

$$Y_1=-(\Delta''+2\Delta^2)/4\Delta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (175)$$

Thus if Δ is known, Y_1 may be determined explicitly and without quadratures. Substituting this value of Y_1 in (173) similarly determines $f(X)$. The function Δ measures the shape of the bulge which moves forward as the dislocation moves from its stable to its activated position. It is convenient for this bulge to be even in X and limited to the region where X is small. Moreover, we know from (144) and (146) that in the case $f(X)=2$, eqn. (173) has the solutions $Y_1=-1$ and $Y_1+\Delta=3 \operatorname{sech}^2 X-1$, corresponding to $\Delta=3 \operatorname{sech}^2 X$. We are thus led to consider the problem arising from the choice $\Delta=\gamma \operatorname{sech}^2 X$, which leads to a stress field of the form (156).

7.26. *A Model of a Localized Barrier*

Professor Cottrell has suggested to me a class of models* which differ essentially from those already considered. The critical stress P_1 is large, but acts in a very small region, so that the activation energy may be of order mkT even if the applied stress P_0 is much smaller than P_1. As an example we may consider that the active glide plane is threaded by stationary dislocations corresponding to another glide system.

---

* See Cottrell (1952b).

Suppose there are no long range forces between the two sets of dislocations, but that an atomic derangement of high energy is produced in a small region with linear dimensions of order b if the lines of both dislocations pass through this region. As a specific model, we may suppose that the two dislocations are both of pure edge type, and would pass freely through each other, but for the fact that each is dissociated into two partial dislocation (§ 5.21 of Part III). The four partial dislocations all have screw components, and cannot pass through one another pair by pair. The complete dislocations can only pass through one another if their components recombine over a short distance near the crossing point. This idea is due to Heidenreich and Shockley (1948). An activation energy U_0 is needed for this, and in the absence of thermal vibrations a moderate stress P_0 will not appreciably alter the separation of the components of the two dislocations.

Suppose a dislocation is held up in this way at a succession of points separated by distances $2R$. The position of stable equilibrium depends very little on the applied stress P_0. If the activation energy U is supplied at one of the locking points, the dislocation at this point moves forward to a position of unstable equilibrium, a distance θb ahead, where θ is of order unity. The loops of dislocation joining this point to its neighbours each move forward by an average distance $\frac{1}{2}\theta b$, and the applied stress does work $2\theta b^2RP_0$. The activation energy U for the stressed system is thus

$$U=U_0-2\theta b^2RP_0. \qquad \qquad (176)$$

If creep is first observed when $U=mkT$, we have a strength given by

$$P_0=U_0/2\theta b^2R-(mk/2\theta b^2R)T. \qquad \qquad (177)$$

If we compare two similar systems with the same value of U_0, but differing spacings R and R' of the locking dislocations, they will begin to creep at the same temperature under different stresses P_0 and P_0', where

$$RP_0=R'P_0'. \qquad \qquad (178)$$

When the linear relation (176) holds between U and P_0, incremental creep tests (§ 7.2) give directly the quantity $dU/dP_0=-2\theta b^2R$. It follows from (178) that in two such tests at the same temperature on different specimens the quantities $P_0\,dU/dP_0$ and $P_0'\,dU'/dP_0'$ must have the same value. Wyatt's observations (1950) satisfy this relation well.

7.3. THE AVERAGE STRESS ON A DISLOCATION WHICH IS NEARLY STRAIGHT

Consider a dislocation lying in a medium in which the internal stress varies from place to place about a mean value zero. Suppose the average amplitude of the varying stress is P_i, while the distance between neighbouring regions of maximum stress is Λ. The stress P_i will bend the dislocation into arcs of radius ρ given by

$$\rho=E/bP_i.$$

If $\rho \ll \Lambda$, the internal stresses can bend the dislocation so that it lies almost entirely in regions of low potential energy where the internal stress is much less than P_i. If an external stress is applied, the dislocation moves forward by a process in which individual loops of length Λ move forward by a distance Λ from one potential trough to the next. The analysis of § 7.22 was developed for this process. No quantitative analysis has been attempted for the case $\rho \simeq \Lambda$, which seems to represent a precipitation-hardened alloy treated to develop maximum hardness. We now consider the case $\rho \gg \Lambda$, which may be written $E \gg bP_i\Lambda$. Since E is of order $\frac{1}{2}b^2\mu$, this inequality may also be written

$$\mu/P_i \gg 2\Lambda/b. \qquad (179)$$

When (179) is satisfied, the internal stresses are not large enough, in comparison with the tension of a dislocation, to force the dislocation into the regions of low stress. Lengths of the dislocation large in comparison with Λ run almost straight across the crystal, being bent by individual regions of stress into arcs with radii of curvature $\rho = E/bP_i$. The amplitude of the wavy curve into which the dislocation is bent is of order $\Lambda^2/\rho \simeq 2\Lambda^2 P_i/b\mu$.

If the dislocation showed no large scale deviations from straightness it would move under the influence of any external stress, however small. For a dislocation of a length L which is much greater than Λ is composed of L/Λ elements, on each of which the internal stress exerts a force of order of magnitude $bP_i\Lambda$ and of random sign. The resultant force is of order $bP_i\Lambda(L/\Lambda)^{1/2}$, while the force due to an external stress P is bPL, which can always be made larger than $bP_i\Lambda(L/\Lambda)^{1/2}$ by taking L large enough.

In fact, we must not assume that the mean path of the dislocation will not depart from a straight line, however large L becomes. We can find a length L so large that a dislocation of this length is effectively flexible under the internal stresses. The motion of a dislocation under an external stress takes place by the movement of individual segments of this length L. No rigorous way of estimating L has been discovered. Two plausible methods have been suggested, and these lead to essentially different results. Masing (1948) has given an exact treatment for an extremely simplified model.

Mott and Nabarro (1948) pointed out that the mean force on a length L, equal to $bP_i(\Lambda/L)^{1/2}$, would bend the mean position of the dislocation into a curve of radius

$$\rho_L = \frac{E}{bP_i}\left(\frac{L}{\Lambda}\right)^{1/2} \simeq \frac{b\mu}{2P_i}\left(\frac{L}{\Lambda}\right)^{1/2}. \qquad (180)$$

If L is so large that this radius ρ_L is comparable with L, we may say that the dislocation is effectively flexible under the internal stress. The condition for this is

$$L = \frac{E^2}{b^2 P_i^2 \Lambda} \simeq \frac{b^2\mu^2}{4P_i^2\Lambda}, \qquad (181)$$

and the average stress on a length L is

$$\left(\frac{\Lambda}{L}\right)^{1/2} P_i = \frac{\Lambda b P_i^2}{E} \simeq \frac{2\Lambda P_i^2}{b\mu} . \quad . \quad . \quad . \quad . \quad . \quad (182)$$

The same argument may be put in another form. Each segment of length Λ is bent into an arc of radius E/bP_i. Its ends are therefore inclined at an angle $bP_i\Lambda/E$. The average angle between the mean directions of the dislocation at points L apart is $(L/\Lambda)^{1/2}bP_i\Lambda/E$. If this is of order one radian, we again have the condition (181).

Mott (1952 b) has suggested another way of estimating the length L of the loop of dislocation which moves independently. He assumes that neighbouring positions of equilibrium for such a loop will be a distance Λ, apart near the middle of the loop, while coinciding at the ends of the loop. We may take these positions to be circular arcs bulging a distance $\frac{1}{2}\Lambda$ to either side of the chord of length L joining their ends. Then $L^2=4\rho_L\Lambda$, giving

$$L = \left(\frac{4E}{bP_i}\right)^{2/3} \Lambda^{1/3} \simeq \left(\frac{2b\mu}{P_i}\right)^{2/3} \Lambda^{1/3}, \quad . \quad . \quad . \quad . \quad (183)$$

and an average stress

$$\left(\frac{\Lambda}{L}\right)^{1/2} P_i = \left(\frac{\Lambda b P_i}{4E}\right)^{1/3} P_i \simeq \left(\frac{\Lambda P_i}{2b\mu}\right)^{1/3} P_i . \quad . \quad . \quad . \quad (184)$$

PART III. EFFECT OF THE CRYSTAL STRUCTURE

§ 1. ELASTIC DISLOCATIONS AND CRYSTAL DISLOCATIONS

The theory of elastic dislocations can be applied to determine the stresses far from a dislocation line, and to establish general theorems, but there are important differences between elastic dislocations and dislocations in crystal lattices. The elastic dislocation is made in a multiply connected body containing certain fixed holes. If we try to continue the solution into the interior of these holes the strain becomes very large. For a given radius of hole the strain in the medium can be kept less than some predetermined small value $e_{\max}$ by reducing the vector $\mathbf{b}$. In the crystal dislocation, an analysis of atomic displacements is required at distances of the order of one interatomic spacing from the dislocation line, while the vector $\mathbf{b}$ must be a lattice vector. Estimates of the strain are therefore required where the strain is no longer small. It is an essential part of the picture of crystal dislocations that for these finite strains Hooke's law is no longer valid, and in particular that a relative displacement of two layers of atoms by a distance $\mathbf{b}$ produces zero restoring force. Because of this, and because the toroidal holes of the elastic theory are reduced to lines threading the lattice, a crystal dislocation is mobile.

In defining the Burgers vector **b** of a crystal dislocation, it is convenient to use the method of counting steps round a closed circuit which was described at the end of § 2.3 of Part II, instead of the circuit integral (17) at the beginning of that section. In making this count, no attention is paid to the fact that the steps are not all precisely parallel. The circuit must lie entirely in regions of moderate or small strain where local crystal axes can be identified. The local axes at one point of the circuit may be rotated with respect to those at another point, and the labelling of the individual axes is determined by continuous comparison along the circuit. The result is expressed in lattice vectors of the lattice at the arbitrary initial point. It will be seen that the residue of steps is $-\mathbf{b}$, for an increase in strain increases the length of each step, and decreases the number of steps. This method, which is based on the fact that the displacement in a crystal lattice is only well defined at each lattice point, is not only more convenient, but also more consistent. In an elastic dislocation the idea of a continuous distribution of dislocation lines is meaningless, as each line must be enclosed in a hole, but a continuous distribution of dislocations arises in some theories of crystal dislocations (e.g. § 2.1). For such dislocations the value of the integral (17) is not the same for all circuits enclosing the ‘ middle ’ of the dislocation, whereas the count has always a residue of a whole lattice vector, equal to the value of the integral taken round a very large circuit. Frank (1951 a) and Read and Shockley (1952) have given detailed discussions of the definition of the Burgers vector of a crystal dislocation.

The general connection between elastic dislocations of the first three orders and crystal dislocations is apparent, and we have seen that dislocations of the last three orders do not lead to simple glide when they move across the body. What are the properties of a crystalline body which contains a dislocation of the latter group ? The crystal lattices on either side of the cut surface are inclined to one another. The surface is thus a definite discontinuity in the structure. For isotropic media, and also for dislocations of the first three orders in crystalline media, the surface is arbitrary so long as its boundary remains fixed, but this result is no longer true for dislocations of the last three orders in a crystal. For example, if the ring of fig. 1 is crystalline, its final state will depend on the orientation of the line AA′ with respect to the crystal axes. Just as the rotation of Σ which generates the dislocations of orders 4, 5 and 6 may be regarded as composed of displacements of its elements proportional to their distances from the origin, so the dislocations may be regarded as continuous distributions of dislocations of order 1, 2 and 3 over Σ. Thus it is apparent from eqn. (28) of Part II that the dislocation d_{xy} is closely related to a uniform distribution of infinitesimal dislocations b_x and b_y. This analysis takes a slightly different form in the crystal lattice, the continuous distribution of infinitesimal dislocations being replaced by a grid of localized dislocation lines, each of strength one lattice vector (cf. § 2.31).

Bilby (1950 c) has published some instructive photographs of models of dislocations in a simple cubic lattice. We shall discuss later (§ 5.21) the relation between the dislocations in a two dimensional raft of bubbles and the dislocations of a close packed crystal.

According to § 2.1 of Part II, the strain at a distance r from a line dislocation in an elastic medium obeying Hooke's law is of order r^{-1}, and therefore infinite at $r=0$. If the energy of the dislocation is expressed as an integral over r, this integral diverges at both limits. The divergence at infinity is inevitable; that at the origin arises solely from the mathematical approximation of assuming Hooke's law to hold for large strains. In fact, if one plane slides over another, the shear stress between them must be periodic with period b. Approximate solutions have been derived for the displacements in a body containing a dislocation on the assumption that the shear stress across the glide plane in any region is a sinusoidal function of the amount of glide which has occurred in that region. The approximations are therefore only valid if the relative displacement of points above and below the glide plane changes by amounts much less than b in passing from one pair of atoms to a neighbour. Two types of approximation have been based on this idea. In the first, due to Peierls (1940), the material on either side of the glide plane is assumed to behave as an isotropic elastic continuum. The second type of approximation, due to Frenkel and Kontorova (1938), assumes the material below the glide plane to be rigid, while that above the glide plane consists of a single row or sheet of atoms connected by springs. The second model is only of illustrative value in the study of slip, but its properties are more easily studied, and it has other important applications. We discuss these approximations in §§ 2 and 3, while § 4 deals with the more delicate problems which arise as soon as we take account of the atomic structure of the surfaces which face one another across the glide plane. Finally, § 5 treats problems which depend on the detailed structure of the lattice under consideration.

§ 2. THE APPROXIMATION OF PEIERLS

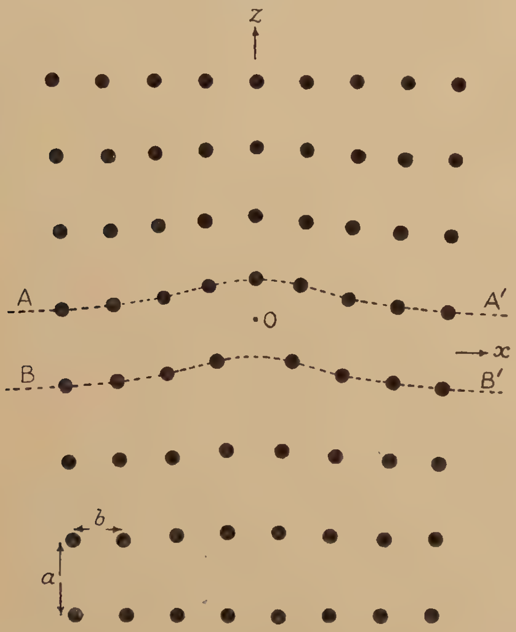
2.1. THE SOLUTION FOR A SINGLE DISLOCATION

Figure 33 represents a crystal containing an edge dislocation lying along the axis Oy. The glide plane $z=0$ is taken midway between the sheets of atoms A and B which slip over one another. The natural spacings of atoms in the x and z directions are b and a respectively, and the material is considered to be elastically isotropic. In some applications of the model we shall assume $a=b$. This does not materially affect the analysis. The atoms of A in the neighbourhood of O are spaced at distances less than b. They are subject to two forces. The elastic reaction of the material above A tends to increase the spacing to b, while the attraction of the atoms in B tends to reduce the spacing still further.

We calculate these forces separately as functions of the positions of the atoms, and determine the positions of the atoms by requiring the two forces to balance at all points of the glide plane.

Since the dislocation is long and straight, the problem is one of plane strain. Let the displacements of an atom of A be (u, w), while those of a neighbouring atom in B are $(\bar{u}, \bar{w})$. The origins of these displacements are chosen so that when all the displacements vanish the atoms in A and B are their correct distance a apart in the z direction, but are $\frac{1}{2}b$ out of step in the x direction, as is shown near the point O. Then it is easily seen that $\bar{u}=-u$, $\bar{w}=w$, so that A and B, though curved by the presence of the dislocation, still fit together without any stress p_{zz} between them.

Fig. 33



Peierls's model of a dislocation. The material above AA′ and below BB′ is regarded as forming an elastic continuum. The force between the rows AA′ and BB′ is a periodic function of the displacement.

To calculate the elastic force, we consider A and the rows above to form an elastic continuum, to the free surface of which is applied a tangential stress p_{xz}. Then formal elastic theory leads to the relation

$$p_{xz}=\frac{\mu}{\pi(1-\sigma)}\int_{-\infty}^{\infty}\frac{du(x')}{dx'}\frac{dx'}{x'-x}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the Cauchy principal value of the integral is taken.

Eshelby (1949 a) has given a direct proof of (1) using the known results for a dislocation in a continuum. The variation of $u(x')$ with position x' is considered to be caused by a continuous distribution of elementary edge dislocations along the x axis, with density $b'(x')\,dx'$. Since the relative displacement of the two surfaces is $2u$, we have $b'(x')=-2\,du/dx'$. The shear stress which this infinitesimal dislocation produces at the point x on the glide plane is given by (9) of Part II as $b'\mu/2\pi(1-\sigma)(x-x')$, and (1) follows by integrating over all x'.

The stress given by eqn. (1) is to be equated to that caused by the atoms in B, which we assume to be a sinusoidal function of the relative displacement $u-\bar{u}=2u$, with period b. The amplitude of the function is determined by the condition that a small relative shift of A and B from their equilibrium positions produces a restoring stress equal to that given by Hooke's law for the same shear. This leads to

$$p_{xz}=-\frac{b\mu}{2\pi a}\sin 2\pi\phi, \qquad \qquad (2)$$

where $\phi=2u/b$.

Comparing eqns. (1) and (2), we obtain the integral equation

$$\int_{-\infty}^{\infty}\frac{d\phi(x')}{dx'}\,\frac{dx'}{x-x'}=\frac{1-\sigma}{a}\sin 2\pi\phi. \qquad \qquad (3)$$

The solution corresponding to fig. 36 is

$$\left.\begin{aligned}\phi&=-\frac{1}{\pi}\tan^{-1}\frac{2(1-\sigma)x}{a},\\ \text{or}\qquad u&=-\frac{b}{2\pi}\tan^{-1}\frac{x}{\zeta},\end{aligned}\right\} \qquad \qquad (4)$$

where $\zeta=a/2(1-\sigma)$.

The width of the dislocation is small, for the displacement falls to half its extreme value at a distance ζ from the centre. The original assumption that the dislocation is spread over a large number of atoms has thus led to a contradiction, and, if the sinusoidal law of force (2) and the neglect of stresses p_{zz} are valid approximations, a dislocation must in fact be only a few atoms wide. We shall discuss this question again in § 2.4.

The displacements at all points of the crystal above A may be calculated (Leibfried and Lücke 1949). The singularity which in the pure elastic theory arises at the origin is now found at a distance ζ below A outside the upper half crystal, while the displacements in the lower half crystal have a corresponding singularity at a height ζ above B. We may regard the displacements in A as caused by an elastic dislocation passing through the point $(0,\ \frac{1}{2}a-\zeta)$, together with a uniform dilatation along this line, or alternatively we may regard the displacements at *all* points of the crystal as derived from elastic dislocations spread continuously over the x axis with density $b\zeta/\pi(x^2+\zeta^2)$.

A screw dislocation is not confined to a single glide plane, and the assumption of a special law of force between a particular pair of planes of atoms does not at first sight seem to provide a satisfactory model.

However, Eshelby (1949 b) has pointed out that if this assumption is made, the analysis proceeds exactly as for the edge dislocation, except that σ in the formulae is replaced by zero. The corresponding value of ζ is $\frac{1}{2}a$, so that the stresses on *both* sides of the assumed glide plane coincide with those produced by an elastic dislocation at the origin, and are independent of the initial choice of glide plane.

If the crystal is not infinite in the z direction, but is bounded by the planes $z=\pm\frac{1}{2}D$, the elastic solution for a screw dislocation may be found by the method of images. Leibfried and Dietze (1949) have shown that if a dislocation lies in the middle of the plate, and we write $\frac{1}{2}D'=\frac{1}{2}(D-b)$ for the width of the portion of the plate lying above $z=\frac{1}{2}b$, eqn. (1) is replaced by

$$p_{yz}=\frac{\mu}{D'}\int_{-\infty}^{\infty}\frac{dv(x')}{dx'}\frac{dx'}{\sinh\left[\pi(x'-x)/D'\right]}. \quad . \quad . \quad . \quad . \qquad (5)$$

Just as in the case of the infinite crystal, the corresponding Peierls equation is satisfied by the displacements produced by an elastic dislocation a suitable distance $\zeta(D/b)$ below the plane $z=\frac{1}{2}b$. The total energy is finite.

For the edge dislocation, the energy of misfit is

$$\frac{b^2\mu}{4\pi^2a}\int_{-\infty}^{\infty}(1+\cos 2\pi\phi)\,dx=\frac{b^2\mu}{4\pi(1-\sigma)}. \quad . \quad . \quad . \quad . \qquad (6)$$

The elastic strain energy is $\int p_{xz}u\,dx$. If this integral is taken between infinite limits it diverges. Between limits $\pm R$, it yields

$$\frac{b^2\mu}{2\pi^2(1-\sigma)}\int_0^{R/\zeta}\frac{y\tan^{-1}y\,dy}{1+y^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

which for large R is asymptotic to $b^2\mu\ln(R/2\zeta)/4\pi(1-\sigma)$.

Lomer (1952) has pointed out that the energy is reduced if the stresses are relaxed across the surface of a cylinder of radius R. The stresses corresponding to this relaxation are those derived from the stress function χ^1 given by (52) of Part II. The reduction in energy is easily obtained by integrating along the line $\theta=0$ as

$$\tfrac{1}{2}b\int_0^R p_{r\theta}\,dr=\frac{b^2\mu}{8\pi(1-\sigma)}.$$

2.2. THE SOLUTION FOR A PAIR OF DISLOCATIONS

A single straight dislocation cannot be formed within a crystal by localized glide motions. In Orowan's original picture (fig. 4), glide is initiated by the formation of a closed loop of dislocation. We take as a model of this process the formation of a pair of edge dislocations of opposite sign on the same glide plane. This system cannot be in stable equilibrium, but if a shear stress is applied to the crystal it is possible (Nabarro 1947) to find a position of unstable equilibrium in which the attraction between the dislocations balances the force with which the applied stress tends to separate them. The energy of this unstable state, referred to the homogeneously sheared crystal as zero, is the activation energy for the initiation of slip. The configuration is shown in fig. 34.

It is now convenient to measure displacements in the rows A and B from zeros such that $\phi=0$ represents perfect fit. This changes the sign of the sine function in eqns. (2) and (3). Consider the function

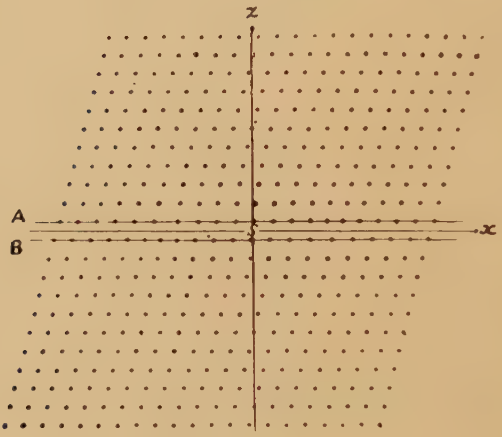
$$\phi=\frac{1}{\pi}\cot^{-1}\left(\frac{x^2\sin\theta}{4\zeta^2}-\cot\theta\right)+\frac{\theta}{4\pi}, \quad . \quad . \quad . \quad . \quad (8)$$

where θ is a constant.

Substituting in eqn. (1), we find that the tangential stress required to maintain this displacement is

$$\frac{b\mu}{2\pi a}(\sin 2\pi\phi-\sin\tfrac{1}{2}\theta),$$

Fig. 34



A pair of dislocations in a sheared crystal. The angle of shear shown is ten times as great as that required to maintain equilibrium in the planes A and B.

whereas the stress produced by the atoms in the other half of the crystal is

$$\frac{b\mu}{2\pi a}\sin 2\pi\phi.$$

The actual stress exceeds that required to maintain equilibrium by a constant quantity $(b\mu/2\pi a)\sin\frac{1}{2}\theta$. The upper and lower halves of the crystal can only remain in equilibrium if external stresses of this amount are applied to them. This stress would produce a constant relative displacement of the planes A and B given by $\phi=\theta/4\pi$, which agrees with the strain given by eqn. (8) for large values of x. In practical cases the applied stress P is usually much less than μ, and θ is therefore small.

For small θ, the two dislocations are widely separated. We may define their centres as the points $x=\pm 2\zeta/\theta$, and the distance between them $w=2x$ is given by

$$w=2a/(1-\sigma)\theta,$$

while

$$P=b\mu\theta/4\pi a,$$

leading to

$$P=b\mu/2\pi(1-\sigma)w.$$

Figure 34 corresponds to $\theta=\pi/10$.

These results for small θ agree with those given by elastic theory. The stress P exerts on each dislocation a force $Pb=b^2\mu\theta/4\pi a$, while the attraction between the dislocations is $b^2\mu/2\pi(1-\sigma)w$, which is also equal to $b^2\mu\theta/4\pi a$.

The activation energy to form this pair of dislocations is the increase in the elastic energy of the two half crystals, plus the increase in the energy of misfit between A and B, minus the work done by the external forces.

If θ is small, the activation energy for unit length of the dislocation pair is $b^2\mu \ln(2/\theta)/2\pi(1-\sigma)$. Adding the work bPw done by the external forces, we find the energy associated with each dislocation to be

$$\frac{b^2\mu}{4\pi(1-\sigma)}\left(1+\ln\frac{w}{2\zeta}\right).$$

This is the work done by the stresses in the elastic solution when the range of integration is cut off at a distance $2\zeta/e$ from the centre of each dislocation. This cut-off radius agrees with that obtained for a single dislocation. For a screw dislocation, the elastic stresses lead to a similar expression in which the factor $(1-\sigma)$ is replaced by unity. Dean and Mann (1949) have treated the problem of two edge dislocations by the methods of elasticity theory. They neglect the boundary condition at infinity, and their result therefore depends on the shape of the outer boundary.

2.3. ARRAYS OF DISLOCATIONS

van der Merwe (1950) has extended the Peierls approximation to three problems involving uniformly spaced rows of dislocations. We outline his treatment in §§ 2.31, 2.32 and 2.33. Koehler (1941) has calculated the energies of lattices of positive and negative dislocations, assuming that the elastic solution holds at all points distant more than r_0 from a dislocation, while the energy stored within r_0 is independent of the presence of other dislocations. The lattices he considers contain pairs of positive and negative edge dislocations with their joins lying perpendicular to the glide plane. The energy is obtained as a lattice sum using potentials similar to (65) of Part II. A pair of dislocations is taken as reference unit, and the potential of interaction between this and a similar unit at a distance r is of order r^{-2}. This is summed over all units lying on the axis of the reference unit. The interaction between the reference unit and all units lying on a parallel line at a distance a diminishes exponentially with a according to a formula of the type (92), (98) or (99) of Part II, and the summation over all lines is rapidly convergent, as in the sums of § 5.3 of Part II. The energy W for each pair of dislocations is of the form

$$W=\frac{b^2\mu}{2\pi(1-\sigma)}\left(\ln\frac{d}{b}+\gamma\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where d is the separation of dislocations in a pair, and γ is a constant depending on the lattice structure and on the law of force assumed to hold in the core of each dislocation. Koehler's formulae are incorrect in detail, as his expression for the interaction energy of a pair of edge dislocations differs from (65) of Part II by numerical factors.

2.31. *Edge Dislocations on the same Glide Plane*

van der Merwe first considered a row of edge dislocations on the same glide plane in a simple cubic lattice. This configuration occurs between the bent lamellae of fig. 2, or in the interface between a thin sheet of crystal and an oriented layer of equal thickness deposited on it. The initial stages of the analysis are similar to those of § 2.1. The natural spacings of atoms in the layers A and B of fig. 33 are no longer equal, but are b_{A} and b_{B} respectively, having a vernier P such that $(P+1)b_{\mathrm{A}}=Pb_{\mathrm{B}}$. It is convenient to refer the positions of atoms in each lattice to a symmetrical reference lattice C having a spacing b_{C} such that $(P+1)b_{\mathrm{A}}=(P+\frac{1}{2})b_{\mathrm{C}}=Pb_{\mathrm{B}}=p$. Then each atom of B corresponds through an atom of C with a unique atom of A, and the force between atoms of A and B is a periodic function of their relative displacement, with period b_{C}. The displacement itself has period p, and the analysis is therefore conducted with Fourier series instead of Fourier integrals. The governing equation is

$$2\pi\frac{d\phi}{dX}-1=-\frac{1}{2\pi\tan\delta}\int_0^{\pi}[\sin 2\pi\phi(X+t)-\sin 2\pi\phi(X-t)]\cot\tfrac{1}{2}t\,dt, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where

$$\phi=2u'/b_{\mathrm{C}}$$

$$X=2\pi x/p$$

and

$$\tan\delta=\pi\mu/(P+\tfrac{1}{2})\bar{\mu}(1-\sigma).$$

Here $\bar{\mu}$ is an elastic constant approximately equal to μ, and u' is the displacement measured from the reference lattice. The solution of (10) is shown to be

$$\phi(X)=\tfrac{1}{2}+\frac{1}{\pi}\tan^{-1}(\tan\tfrac{1}{2}\delta\tan\tfrac{1}{2}X). \quad . \quad . \quad . \quad . \quad (11)$$

2.32. *Screw Dislocations Forming a Grain Boundary*

The resolution into two grids of screw dislocations of the boundary between two grains which have been rotated about the normal to a common low index plane is illustrated in fig. 24. When the angle of rotation $\pm\frac{1}{2}\psi$ about this normal is small, the only important effect of the interaction between the two grids is to reduce to zero the shear at points far from the interface, and we shall consider the screws along the z axis of fig. 24 (b) separately. Their separation is $p=\frac{1}{2}b \operatorname{cosec} \frac{1}{2}\psi$, and the period of the force of interaction between the adjoining surface is $b \sec \frac{1}{2}\psi$. Assuming that the force is derived from a nearest neighbour interaction, its maximum potential is independent of ψ, and this leads to a force law

$$p_{yz}=\frac{\mu_0 \cos \frac{1}{2}\psi}{2\pi} \sin\left(\frac{2\pi v \cos \frac{1}{2}\psi}{b}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The governing equation reduces to the form (10), with the solution (11). From this solution the surface energy $\tau(\psi)$ of the boundary is calculated, the figure first obtained being doubled to allow for the second set of dislocations. The result is

$$\tau(\psi)=\frac{b\mu}{4\pi^2}\left\{1+\gamma-(1+\gamma^2)^{1/2}-\gamma \ln\left[2\gamma\{(1+\gamma^2)^{1/2}-\gamma\}\right]\right\}, \quad . \quad . \quad (13)$$

where $\gamma=2\pi\mu\mu_0^{-1} \tan \frac{1}{2}\psi \sec \frac{1}{2}\psi$.

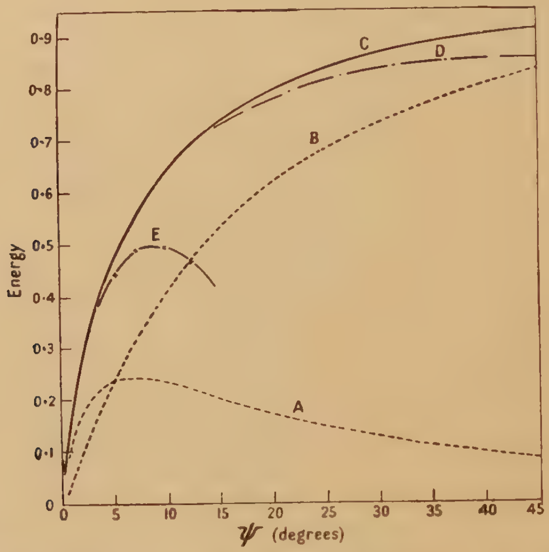
For small ψ this is of the expected form (cf. (95) of Part II). If we attempt to extrapolate (13) to larger values of ψ, we find that it does not have the necessary property of symmetry about $\psi=\frac{1}{4}\pi$, which is the greatest possible relative rotation of two simple square plane lattices. This happens because (12) represents only the interaction between those rows of atoms above and below the boundary in fig. 24 which are parallel when $\psi=0$. When ψ lies between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, it is rows which were originally perpendicular which tend to align. van der Merwe proposed the interpolation formula $\tau_s(\psi)=\tau(\psi)+\tau(\frac{1}{2}\pi-\psi)-\tau(\frac{1}{2}\pi)$, and the two formulae are plotted in fig. 35.

When $\psi=\frac{1}{2}\pi$, all the atoms of one lattice coincide with atoms of the other lattice. When $\tan \frac{1}{2}\psi$ is rational, say m/n, some atoms again come into coincidence, the atom originally at (nb, mb) falling on that originally at (nb, $-mb$), and this may lead to a reduction in $\tau(\psi)$ for this value of ψ. The following argument (based on a suggestion by Dr. G. A. P. Wyllie) shows that the reduction is likely to be small. For any fixed value of ψ, the energy of the surface of misfit may be varied by a rigid translation of the two half crystals. During this translation the grids of screw dislocations move rigidly across the interface. The energy of each dislocation varies as it moves with respect to the atomic lattice, but the calculation of § 4.1 shows that the variation is small in a metallic lattice. In this case an irrational value of $\tan \frac{1}{2}\psi$, which leads to a distribution of dislocation energies about their mean, will not lead to a significantly greater surface energy than a neighbouring rational value for which a suitable translation allows many dislocations to find their positions of lowest energy. The surface energy is abnormally low only for values of ψ close to those which make *all* the atoms on one side of the interface correspond with atoms on the other side because we consider only nearest neighbour interactions across the interface. If an atom A_0 on one side is placed above an atom B_0 on the other side, the energy is low only if ψ is so chosen that a next neighbouring atom A_1 lies very nearly above some nearest neighbour B_1 of B_0. Thus the $\tau(\psi)$ curve for a twist boundary, unlike that for a tilt boundary (§ 5.23 of Part II), should not be markedly cusped. Against this, the experiments of Wilman (1951) indicate a marked preference for certain rational values of tan $\frac{1}{2}\psi$.

2.33. *Edge Dislocations Forming a Grain Boundary*

The analysis of a wall of edge dislocations forming the grain boundary of fig. 21 is similar to that of the preceding sections. The elastic constant corresponding to μ_0 of (12) is shown to be $\mu/(1-2\sigma)$, and the width of the dislocation, measured along the line joining two dislocations, is less than b. This means effectively that the solution reduces to Burgers's solution in which Hooke's law holds everywhere except in the neighbourhood of a dislocation, and in this neighbourhood the approximation breaks down. The results, however, illustrate an important effect, which is shown still more clearly by the calculations for a twist boundary (fig. 35). Suppose that in the approximate formula $\tau(\psi)=\tau_0\psi \ln (\psi_0/\psi)$ we choose ψ_0 in order to obtain the best fit to (13) for small values of ψ. Then for larger values of ψ the energy given by (13) is greater than that given by the approximate formula. We shall discuss the interpretation of this effect in § 2.42.

Fig. 35



Energy of a twist boundary (after van der Merwe). A : Elastic energy. B : Energy of misfit. C : Total energy. D : Symmetrized formula for total energy. E : Total energy according to a formula $\tau=\tau_0\psi \ln (\psi_0/\psi)$ with ψ_0 adjusted to give a good fit for small ψ.

2.4. DEVIATIONS FROM A SINUSOIDAL LAW OF FORCE

If one plane of atoms is laid upon another, its positions of equilibrium are separated by lattice vectors b. It often happens that between two neighbouring positions of equilibrium there is another position of stable mechanical equilibrium corresponding to a twinned or faulted structure. The curve representing the potential energy of the two layers as a function of their relative displacement then has subsidiary minima between the principal minima, which are separated by distances b. Even if there are no minima, the maxima may lie much lower than those given by the sinusoidal law of force (2). Zener (1948 b) has pointed out that the common occurrence of phase changes from close packed to body-centred cubic structures in metals indicates that the maximum restoring force and potential energy are usually much less than (2) would require.

If the potential maximum is very broad and flat, the energy of misfit V between two surfaces is much less than that corresponding to Hooke's law and a sinusoidal law of force, and is almost independent of the relative displacement u (§ 2.1) except when u/b is small. If the width of the dislocation is l, the surface energy is of order lV, while the elastic energy is of order $b^3\mu/l$. The total energy is thus

$$u=lV+b^3\mu/l, \qquad (14)$$

which is smallest when

$$l=b(b\mu/V)^{1/2}, \qquad (15)$$

corresponding to

$$u=2b(b\mu V)^{1/2}.$$

The width of the dislocation is proportional to $V^{-1/2}$. Since the curvature near the potential minima is fixed by Hooke's law, it is easy to see that the maximum shear stress is of order $p_{\mathrm{m}}=(\mu V/b)^{1/2}$, and is attained at a distance of order $u_{\mathrm{m}}=(bV/\mu)^{1/2}$ from the position of stable equilibrium. We note that $lp_{\mathrm{m}}=b\mu$ and $lu_{\mathrm{m}}=b^2$.

It is not in general possible to solve in closed form the equation which replaces (3) when the sinusoidal law of force is replaced by an arbitrary law.

2.41. *The Method of Foreman, Jaswon and Wood*

Foreman, Jaswon and Wood (1951) have given a parametric solution for a family of laws of force. We use a notation which corresponds to that of § 2.1, except that the distance a of that section is assumed equal to b. The parameter a_0 determines the law of force, which reduces to (2) for $a_0=1$, while ψ is a parameter in terms of which x, u, and p_{xz} are expressed for each law of force. The solution is

$$\left.\begin{aligned} x&=a_0\zeta\tan\tfrac{1}{2}\psi\\ u&=-(b/4\pi a_0)[a_0\psi+(a_0-1)\sin\psi]\\ p_{xz}&=(\mu/4\pi a_0{}^2)[2(2a_0-1)\sin\psi+(a_0-1)\sin 2\psi]\\ &\quad -\pi\leqslant\psi\leqslant\pi,\quad a_0\geqslant 1. \end{aligned}\right\} \qquad (16)$$

For all values of a_0, the relation between u and p_{xz} agrees with Hooke's law for small strains, but the maximum shear stress p_{m} is less than that given by a sinusoidal law of force. We now define the half width l more precisely as the shortest distance in which u changes by $\frac{1}{4}b$. Values of l and p_{m} are given for various choices of a_0 in the following table :

a_0	1	2	3	4	5	10	∞
$l/2\zeta$	1·000	1·255	1·653	2·076	2·507	4·697	$0{\cdot}442a_0$
$2\pi p_{\mathrm{m}}/\mu$	1·000	0·787	0·594	0·471	0·389	0·207	$2{\cdot}202a_0^{-1}$
Bubble diameter in mm	—	1·9	—	1·3	—	0·7	—
r_0/b	0·56	1·04	1·49	1·93	2·36	4·53	$0{\cdot}43a_0$
ψ_0 (degrees)	45	26	17	13	11	5·5	$58a_0^{-1}$

The product lp_{m} is almost exactly constant. The fourth row of figures in the table gives the diameter of the soap bubbles (Bragg and Nye 1947, Bragg and Lomer 1949, Lomer 1949) which give a two dimensional model with a law of force approximating to that given by (16) with the given value of a_0. According to Lomer, the law of force for copper is well represented by bubbles of diameter 1·2 mm, corresponding to a_0 rather greater than 4. The alkalis are best represented by bubbles about 2 mm in diameter, corresponding to $a_0=2$.

Foreman (unpublished) has proved that the surface energy is independent of the law of force in the glide plane, not only for a law of the type (16), but for any law. We give here a form of proof due to Foreman and Nabarro. Let the potential energy for a displacement u be $V(u)$. Then the total energy in the slip plane is

$$E=\int_{-\infty}^{\infty} V(u)\,dx=[Vx]-\int x\frac{dV}{du}\,du=-\int_{-\infty}^{\infty} x\frac{dV}{du}\frac{du}{dx}\,dx. \quad . \quad (17)$$

The value of $dV/du=2p_{xz}$ is given by (1). Substituting this value in (17), we obtain

$$E=-\frac{2\mu}{\pi(1-\sigma)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x}{x'-x}\,\frac{du'}{dx'}\,\frac{du}{dx}\,dx'\,dx. \quad . \quad . \quad . \quad (18)$$

Interchanging x and x', we have

$$E=-\frac{2\mu}{\pi(1-\sigma)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x'}{x-x}\,\frac{du}{dx}\,\frac{du'}{dx'}\,dx\,dx'. \quad . \quad . \quad . \quad (19)$$

Adding (18) and (19),

$$2E=\frac{2\mu}{\pi(1-\sigma)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{du'}{dx'}\,\frac{du}{dx}\,dx'\,dx=\frac{2\mu}{\pi(1-\sigma)}\left(\frac{b}{2}\right)^2,$$

giving $E=b^2\mu/4\pi(1-\sigma)$ in agreement with (6).

If we apply a similar treatment in the case of a screw dislocation, the solution represents a dislocation extended along the glide plane originally selected. It is no longer true, as it was for the elastic theory and for the sinusoidal law of force, that the dislocation has circular symmetry and all possible directions of motion are equivalent. If two glide planes both containing b are crystallographically equivalent, a screw dislocation lying in one can only turn into the other by first shrinking until it has circular symmetry. This requires an energy of activation.

2.42. *The Structure and Energy of a Tilt Boundary*

According to eqn. (95) of Part II, the energy of a tilt boundary of angle ψ is roughly proportional to $\psi \ln (\psi_0/\psi)$. This energy rises to a maximum when $\psi=\psi_0/e$, and vanishes when $\psi=\psi_0$. If ψ_0 is not a small angle, it is obvious that the approximations have broken down when ψ exceeds ψ_0/e. Read and Shockley (1950) have calculated ψ_0 in terms of the energy of a single dislocation. If the energy of unit length of a dislocation, integrated from the axis out to a radius R (without relaxing the tractions across the cylindrical surface $r=R$) is expressed in the form const. ln (R/r_0), then

$$\psi_0=eb/2\pi r_0. \qquad (20)$$

Their numerical application to the case of Peierls's model seems to contain an error: it follows from §§ 2.1 and 2.2 that for this model $r_0=2\zeta/e=b/e(1-\sigma)$, giving $\psi_0=e^2(1-\sigma)/2\pi=45°$ for $\sigma=\frac{1}{3}$. Lomer (1952) has evaluated r_0 for the more general law of force used by Foreman, Jaswon and Wood. He finds for the energy of unit length

$$E=\frac{b^2\mu}{4\pi(1-\sigma)}\left[\ln\frac{R(1-\sigma)}{a_0 b}+1+\left(\frac{a_0-1}{2a_0}\right)^2\right]. \qquad (21)$$

The corresponding values of r_0/b and ψ_0 are given in the table of § 2.41. For copper ($a_0=4$), ψ_0 is only 13°, while the maximum energy occurs at 5°.

The value of r_0 corresponding to the approximation of van der Merwe (§ 2.33) is somewhat different. If we assume his parameter μ_0 to be equal to μ, we find $r_0=b(1-2\sigma)/2(1-\sigma)$.

It is to be expected that for sufficiently large values of ψ the energy will lie above that given by the simple formula $\psi \ln (\psi_0/\psi)$, as indicated by the calculations of van der Merwe. The approximate formula was derived by integrating the energy density round each dislocation from a distance of order r_0 to a distance of order $b/2\psi$, where the stress field of the next dislocation is of equal importance. If the dislocation is of a width l considerably greater than b, this approximation breaks down when $b/2\psi$ becomes comparable with $\frac{1}{2}l$, that is to say, ψ becomes as great as b/l. For Peierls's approximation this is $\psi=1-\sigma \simeq 40°$: for the law of force (16) with $a_0=4$ it is $\psi=20°$. When ψ is greater than this value, the energy associated with each dislocation is approximately that lying in an ellipse with semi-axes $b/2\psi$ and $\frac{1}{2}l$, which may be split into a core energy C and a strain energy D/ψ. The energy of the grain boundary in this region would thus be of the form $C\psi+D$.

The form $C\psi+D$ is based on the assumption that the length and energy of the core of each dislocation remain unaltered even when the dislocations are separated from each other by distances less than their width. The experiments of Lomer and Nye (1952) on grain boundaries in bubble rafts show that this is not correct: as ψ increases to a value at which the spacing of dislocations is comparable with their length, the dislocations become narrower, so that the region of misfit is concentrated in the grain boundary. That this is bound to happen can be seen theoretically in two ways. Firstly, the width of an isolated dislocation is determined by a balance between the strain energy and the energy of misfit. In a tilt boundary of large angle, the energy of misfit predominates if the dislocations have the same width as when they are isolated. The configuration will therefore adjust itself to reduce the energy of misfit, the corresponding increase in strain energy being less than for an isolated dislocation. Alternatively, we may regard each wide dislocation as being a distribution of infinitesimal dislocations along the glide plane. If the dislocations form a tilt boundary, the same forces which draw the dislocations as a whole into a line tend to draw the infinitesimal components of each dislocation into the same line.

The experiments on a raft of bubbles show that as ψ increases through a fairly small range the form of the grain boundary alters from one in which the misfit is concentrated in strips around the glide planes of individual dislocations to one in which the regions of misfit lie perpendicular to the glide plane. The regions of misfit of individual dislocations rapidly overlap, so that it is impossible to distinguish the individual dislocations, and the structure reduces to two almost perfect grains fitting badly along their boundary. It is probably this region which gives a grain boundary energy almost independent of ψ. A theoretical discussion has been given by Shockley (1952).

2.5. BARRIER PROBLEMS

The calculation of § 5.1 of Part II gives the stress field surrounding a procession of dislocations on the same glide plane when the leading dislocation is held up by a barrier. However, it does not give a detailed analysis on an atomic scale of the stresses near the leading dislocation, nor does it provide any physical model of the barrier. This problem has been treated on the Peierls approximation by Jaswon and Foreman (1952). They treat first the problem of a single dislocation meeting a barrier. Their model is similar to that of Peierls, except that for $x \geqslant s$ the law of force across the glide plane alters, while the elastic properties of the medium on either side of the glide plane remain unaltered. This change in the law of force constitutes a barrier at the point $x=s$ to a dislocation approaching from the left. If the sinusoidal law of force is replaced on the right of the barrier by Hooke's law, we have a fair model of a grain boundary, the slip directions permissible in the grain on the right differing from the slip direction of the dislocation in the grain on the left. The barrier repels the dislocation, which is held at the origin by an applied stress P. The *position of the dislocation* is defined to be the point of maximum misfit, at which $u=0$. If there is a crack along the glide plane to the right of $x=s$, the shear stress vanishes, P is negative and the equilibrium of the dislocation at $x=0$ is unstable.

Equation (1) is now replaced by

$$p_{xz}(x)=\frac{\mu}{\pi(1-\sigma)}\int_{-\infty}^{\infty}\frac{du(x')}{dx'}\,\frac{dx'}{x'-x}+P, \quad . \quad . \quad . \quad . \quad . \quad (22)$$

while (2) is replaced by

$$p_{xz}=\left\{\begin{array}{ll} -\dfrac{\mu}{2\pi}\sin\dfrac{4\pi u}{b} & x<s \\ -2\mu u/b & x\geqslant s \text{ (grain boundary)} \\ 0 & x\geqslant s \text{ (crack).} \end{array}\right\} \quad . \quad . \quad . \quad (23)$$

If the solution $u(x)$ which satisfies (22) and (23) is known, the shear stress P may be found from it by a simple virtual work argument. We consider a solution representing a single dislocation, so that $u(\infty)-u(-\infty)=-\frac{1}{2}b$. Then in a virtual displacement in which the whole pattern of atoms in the planes $z=\pm\frac{1}{2}b$ moves forward in the x direction rigidly by a distance δ, the applied stress P does work $bP\delta$. The strain energy of the elastic continua above and below these planes is unaltered, while the relative displacement of a pair of atoms opposite to one another in the glide plane increases by $-2\delta\, du/dx$. The energy of the interatomic forces in the glide plane thus increases by

$$-2\delta\int_{-\infty}^{\infty} p_{xz}(x)\frac{du}{dx}\,dx.$$

By the principle of virtual work, this is equal to the work done by the external stresses, so that

$$P=-\frac{2}{b}\int_{-\infty}^{\infty} p_{xz}(x)\frac{du}{dx}\,dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

This may be written in the form

$$P=-\frac{2}{b}\int_{u(-\infty)}^{u(\infty)} p_{xz}(u)\,du, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where the integral is taken over the branch of the function $p_{xz}(u)$ determined by the corresponding value of x. Thus we cannot evaluate the integral (25) exactly until $u(x)$ is known.

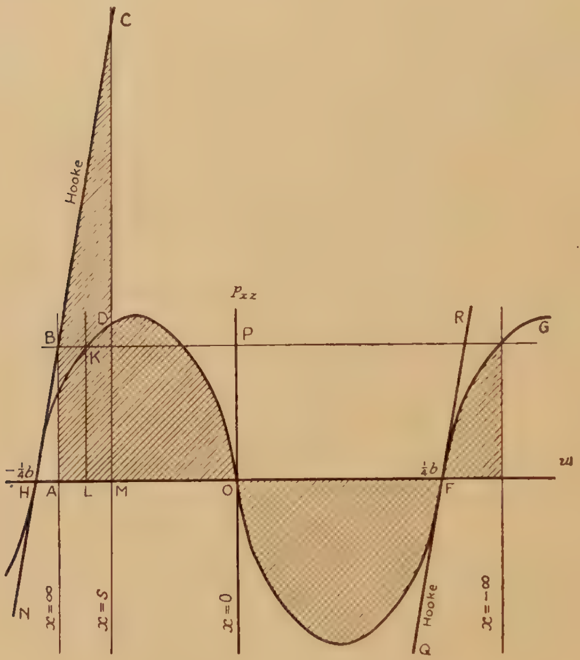
However, if P is small, and the law of force on the right of the barrier does not deviate greatly from the sinusoidal law for small displacements from the position of stable equilibrium, we may obtain a useful approximation. Consider, for example, the case of Hooke's law, illustrated in fig. 36. The value of P is $2b^{-1}$ times the positive area between the u axis and the curve ABCDOFG, shown shaded. Since, for the sinusoidal law, the area under the curve LKDOFG vanishes, the area required is that of the curve ABCDKLA. We note that AB is of order P, while AL is of order P^3. The areas ABKLA and HBKH are thus of order P^4. Since the required area is of order P, we may approximate to it by the area HBCDKH. Calling the difference between the two laws of force (23) $\Delta p(u)$, we may write this result

$$bP=2\int_{-\frac{1}{4}b}^{u(s)}\Delta p(u)\,du. \qquad (26)$$

For Hooke's law,

$$\Delta p(u)\simeq 16\pi^2\mu(u+\tfrac{1}{4}b)^3/3b^3. \qquad (27)$$

Fig. 36



The sinusoidal law of force HKDOFG approximates to Hooke's law NHC or QFR near the positions of stable equilibrium H and F. The force on the dislocation is proportional to the area ABCDKLA, which for small applied stresses OP is nearly equal to HBCDKH.

If P is small, s is large, and we may assume that $u(s)$ is given closely by the solution (4) which holds in the absence of a barrier. This gives

$$u(s)+\tfrac{1}{4}b\simeq b\zeta/2\pi s=b^2/4\pi(1-\sigma)s. \qquad (28)$$

Substituting (27) and (28) in (26), we obtain

$$P=\mu b^4/96\pi^2(1-\sigma)^4s^4. \qquad (29)$$

As a typical numerical value, $s=5b$ when $P=10^{-5}\mu$.

We may now extend the result to find an approximate solution for a procession of n dislocations held up by a barrier. Our treatment will be valid provided that the distance from the leading dislocation to the barrier is appreciably less than its distance from the second dislocation. According to (29), the repulsion of a barrier falls off as s^{-4}, and our assumption about the spacing of dislocations means that the forces exerted by the barrier on all dislocations other than the first are negligible. In this case, s is given by (29) with P replaced by nP. If the distance between the first and second dislocations is d, the total stress on the second dislocation, which must vanish in equilibrium, is $(n-1)P-b\mu/2\pi(1-\sigma)\,d$. Thus $d=b\mu/2\pi(1-\sigma)(n-1)P$, and the condition $d>s$ which must be satisfied if this simple treatment shall be valid may be written

$$nP<0{\cdot}85\mu. \qquad (30)$$

We have here neglected the difference between n and $n-1$.

If, for the displacements actually occurring in this region, the law of force beyond the barrier does not differ greatly from the sinusoidal law, an approximate solution of (22) and (23) may be found by a perturbation method. We take as zero order solution u_0 the solution (4), and write $u=u_0+u_1$, where u_1 and its derivative are small. This leads to a *linear* integral equation for u_1. Jaswon and Foreman have obtained the general solution of this integral equation in series form, using the sets of functions

$$C_n(x)-(-1)^n=\cos\,(2n\tan^{-1}x)-(-1)^n$$

and

$$S_n(x)=\sin\,(2n\tan^{-1}x).$$

§ 3. THE APPROXIMATION OF FRENKEL AND KONTOROVA

Frenkel and Kontorova considered a mechanism of slip differing from that of dislocations. Their basic slip process is the propagation of a compressional wave along a line of atoms. The atoms in the front portion of the line are still in their original positions, while the atoms of the rear portion have moved forward by one lattice distance b. Formally, this may be considered as the motion along the line of atoms of a small closed loop of dislocation with its Burgers vector $\mathbf{b}$ lying along the line, and the motion of a straight dislocation line may be considered as the resultant of many of Frenkel's processes. The motion of a Frenkel ‘ caterpillar ’ does not relieve a shear stress applied to the crystal, and caterpillars cannot be set in motion by uniform external stresses ; the motion is of the kind described in § 4.1 of Part II as ‘ ineffective ’. The physical process considered is probably not of importance in plasticity, but the mathematical analysis provides a useful simplified model of a dislocation (Dehlinger and Kochendörfer 1940). Paneth (1950) has suggested that this mechanism may sometimes be important in diffusion.

All atoms other than those in the moving chain are considered fixed. Each atom in the moving chain has an energy of interaction with the substrate which is a sinusoidal function of its position x, and there are elastic forces between atoms of the chain proportional to the deviation of the distance between nearest neighbours from its equilibrium value a. The potential energy of the chain is thus of the form

$$A\sum_k(1-\cos 2\pi x_k/a)+\tfrac{1}{2}\alpha\sum_k(x_{k+1}-x_k-a)^2. \quad . \quad . \quad . \quad (31)$$

In the state of lowest energy, each atom lies at the bottom of a potential trough, and we shall assume that the chain is very long and that deviations from this lowest state occur only near the middle of the chain. The summation in (31) may then be taken from $k=-\infty$ to $k=\infty$. Write

$$x_k=(k+\xi_k)a,$$

and (31) becomes

$$U=A\sum_k(1-\cos 2\pi\xi_k)+\tfrac{1}{2}\alpha a^2\sum_k(\xi_{k+1}-\xi_k)^2. \quad . \quad . \quad . \quad (32)$$

The condition of equilibrium for kth atom is $\partial U/\partial\xi_k=0$, or

$$\alpha a^2(\xi_{k+1}-2\xi_k+\xi_{k-1})=2\pi A\sin 2\pi\xi_k. \quad . \quad . \quad . \quad (33)$$

The general solution of this difference equation is not known, although Frank and van der Merwe (1949 a) have illustrated some simple special solutions. In the special cases for which the displacements of neighbouring atoms are nearly equal ($\xi_k-\xi_{k-1}\ll 1$), the difference equation may be replaced by the differential equation

$$\alpha a^2\, d^2\xi/dk^2=2\pi A\sin 2\pi\xi, \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

which integrates to give

$$\left(\frac{d\xi}{dk}\right)^2=\frac{2A}{\alpha a^2}(1-\cos 2\pi\xi). \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

The constant of integration is determined by the fact that for a single dislocation ξ and $d\xi/dk$ both vanish in the unstrained parts of the chain. The solutions obtained by other choices of the constant of integration represent either stable infinite sequences of equally spaced dislocations all of the same sign, or unstable sequences of dislocations of alternating sign. They are expressible in elliptic functions.

Integrating again, we find

$$\tan\tfrac{1}{2}\pi\xi=\exp[\pm\pi(k-\kappa)/l_0], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where

$$l_0{}^2=\alpha a^2/4A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

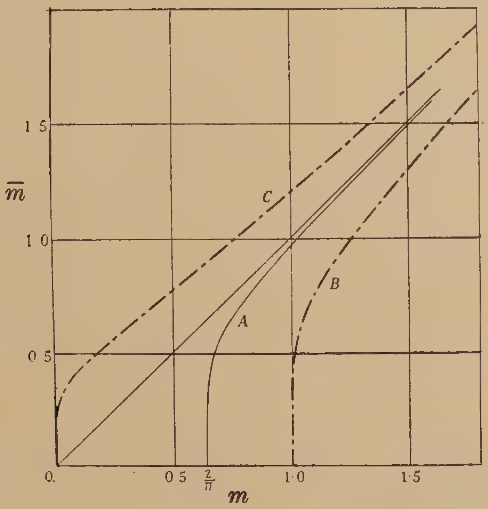
The signs $\pm$ correspond to dislocations of opposite sign, while $k=\kappa$ defines the position of the middle of the dislocation. The solution may be written in the more symmetrical form

$$\tan[\tfrac{1}{2}\pi(\xi-\tfrac{1}{2})]=\tanh[\tfrac{1}{2}\pi(k-\kappa)/l_0].$$

In this one dimensional model the displacement produced by a single dislocation vanishes exponentially at large distances.

Frank and van der Merwe, and also Kochendörfer and Seeger (1950), have extended this treatment to the case in which the natural spacing in the chain b differs from the period of the substrate a. Equation (34) and its solutions are unaltered, but the expression for the tension in the chain is altered. The behaviour of a chain with free ends is shown to depend on the parameter of misfit $m=(b-a)l_0/a$. If the average lattice spacing in the dislocated chain is $\bar{b}$, we define $\bar{m}$ by $\bar{m}=(\bar{b}-a)l_0/a$. A function A is calculated such that in the state of lowest energy $\bar{m}=A(m)$. A second function $B(m)$ is found such that if $\bar{m}<B(m)$ more dislocations form spontaneously without activation energy, and a third function $C(m)$ such that if $\bar{m}>C(m)$ dislocations spontaneously leave the chain. If $A(m)>\bar{m}>B(m)$, the energy is reduced if the number of dislocations in the chain increases, but this increase requires an activation energy. The functions A, B and C are shown in fig. 37.

Fig. 37



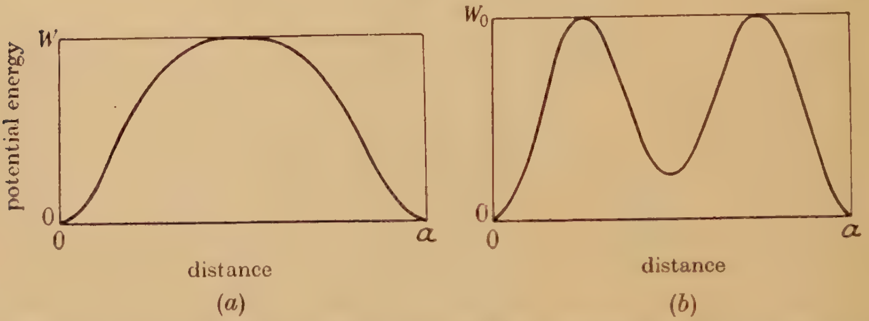
Density of dislocations in a monolayer $\bar{m}$ as a function of the misfit m between the monolayer and the substrate (after Frank and van der Merwe). A: State of lowest energy. B: Below this curve, dislocations form spontaneously in the monolayer. C: Above this curve, dislocations spontaneously leave the monolayer.

The theory has important applications to oriented overgrowths on crystal faces (van der Merwe and Frank 1949, van der Merwe 1949, Frank and van der Merwe 1949 b). However, Smollett and Blackman (1951) have objected to some of the applications of the theory, showing that a monatomic layer strained to the extent which Frank and van der Merwe postulated would in some cases be mechanically unstable. They also disagree with Frank and van der Merwe on the interpretation of some of the experimental data. Further points have been raised by Frank (1951 c), Blackman (1951) and Lucas (1951).

3.1. INTRODUCTION OF A SECOND HARMONIC

Frank and van der Merwe (1949 c) have given an explicit solution of the differential equation for the model of Frenkel and Kontorova when the potential $A \cos 2\pi x/a$ of (31) is replaced by the more general form $A(\cos 2\pi x/a + r^2 \cos 4\pi x/a)$. If $r^2 \leqslant \frac{1}{4}$, the effect of the added term is to sharpen the minima and broaden the maxima of the potential, thus providing a better approximation to the potential in a real crystal. The limiting case of $r^2 = \frac{1}{4}$ is shown in fig. 38 (*a*). Provided that $r^2 \leqslant \frac{1}{4}$, the total range of the potential is still $2A$; if A is determined from the curvature at the potential minimum, its value is $1/(1+4r^2)$ of that corresponding to a simple sinusoidal approximation. The introduction of a second harmonic term with $r^2 \leqslant \frac{1}{4}$ has little effect on the properties of the model.

Fig. 38



(*a*) (*b*)

Effect of a second harmonic term in the potential between two planes of atoms (after Frank and van der Merwe). (*a*) Amplitude of second harmonic a quarter that of fundamental. (*b*) Amplitude of second harmonic four times that of fundamental.

If $r^2 > \frac{1}{4}$, the potential curve has a subsidiary minimum, as shown in fig. 38 (*b*), which corresponds to $r^2 = 4$. The properties of the model are now more complicated, for in addition to the dislocation in which n particles occupy a space of approximately $(n \pm 1)a$, there may be partial dislocations in which the particles to the left of a region of misfit lie near to deep minima, while those on the right lie near to subsidiary minima. The model approximately represents the partial dislocations of § 5.21, although these are not plane deformations and involve shears which the model neglects. The dislocation lies between two close packed planes; the deep minima are sites in the crystal lattice, while the shallow minima are sites in the twinned or faulted lattice.

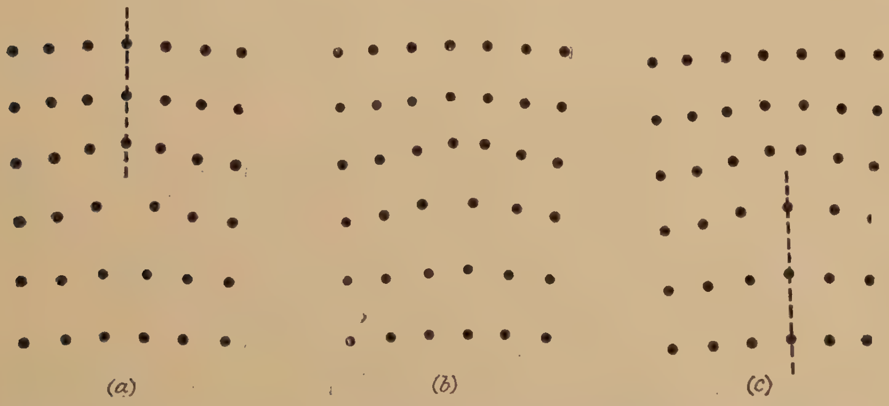
3.2. PERTURBATION THEORY

Seeger and Kochendörfer (1951) have developed a perturbation theory for the model of Frenkel and Kontorova. They apply it to a dynamical problem, and also to the static problem of an array of two or more dislocations not in equilibrium. As they point out, no true equilibrium solution exists for the problems they examine; their solutions found by perturbation theory have the property that they fail to satisfy the governing equation less grossly than do the zero order solutions. These zero order solutions are obtained by joining together portions of different exact solutions. In the cases studied in detail (pairs of dislocations of the same or opposite signs, three dislocations of the same sign), the effect of the perturbation is mainly to increase the distance between the dislocations. It is not clear whether any significance can be attached to the perturbed solutions other than the obvious one that when the dislocations are farther apart they interact less, and therefore the condition of equilibrium can be more nearly satisfied.

§ 4. INTERACTION OF AN EDGE DISLOCATION WITH THE LATTICE POTENTIAL

In treating a straight edge dislocation in a simple cubic lattice, we have hitherto assumed that the dislocation lies symmetrically in the lattice, so that the added half plane of atoms lies midway between two half planes of atoms in the other half crystal. As the dislocation moves through the lattice, it passes through unsymmetrical configurations to a different symmetrical configuration in which one half plane of atoms on the expanded side of the glide plane lies midway between two half planes on the compressed side (fig. 39). Further motion passes through unsymmetrical configurations back to a state equivalent to the original. During this displacement the energy of the dislocation varies, and the dislocation can only move through the lattice if a finite force acts on it.

Fig. 39



Atomic displacements as a dislocation moves through the lattice by a distance $\frac{1}{2}b$. (*a*) Symmetrical position ($\alpha=0$) in which the added half plane ends at the centre of the dislocation. (*b*) Unsymmetrical position ($\alpha=\frac{1}{4}$). (*c*) Symmetrical position ($\alpha=\frac{1}{2}$) in which two rows of atoms above the glide plane match one row below, this row passing through the centre of the dislocation.

In the next section we calculate the variation by a method due to Peierls (1940). The positions of minimum energy for a dislocation in a lattice are straight lines. If the shape of the dislocation departs slightly from a straight line, its energy increases. We estimate this increase in § 4.2.

4.1. CHANGE IN ENERGY OF A STRAIGHT EDGE DISLOCATION AS IT MOVES THROUGH THE LATTICE

The treatment is an extension of that used in § 2.1. We assume that the displacement of any atom in the interface is given by (4) of that section. Instead of calculating the energy of the interface by the integral (6), we take the corresponding sum over the positions of the centres of the atoms above and below the interface. The result now depends on the displacement αb of the centre of the dislocation from the position shown in fig. 39 (*a*). We shall not make the corresponding substitution of a sum for the integral which gives the elastic strain energy. The inconsistency of using displacements calculated for a continuum and then using a summation prevents a clear decision whether this term should be included or not; its inclusion alters only the numerical coefficient in (40). The energy of misfit is

$$v=\frac{b^3\mu}{8\pi^2 a}\sum_{n=-\infty}^{\infty}[1+\cos 2\{\tan^{-1}2(1-\sigma)(\alpha+\tfrac{1}{2}n)b/a\}]. \quad . \quad . \quad (38)$$

This may be transformed by using the relation

$$\sum_{-\infty}^{\infty} f(n)=\sum_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)\cos 2\pi xs\,dx \quad . \quad . \quad . \quad . \quad . \quad (39)$$

into a rapidly convergent series, as was done by Peierls and by Nabarro (1947). However, it may be summed exactly, for we have

$$\frac{4\pi^2(1-\sigma)^2 v}{ab\mu}=\sum_{n=-\infty}^{\infty}\{[a/b(1-\sigma)]^2+[n+2\alpha]^2\}^{-1}.$$

This sum is readily evaluated by considering the poles of the function $\{[a/b(1-\sigma)]^2+[z+2\alpha]^2\}^{-1}\cot \pi z$, and we find

$$v=\frac{b^2\mu}{4\pi(1-\sigma)}\,\frac{\tanh(4\pi\zeta/b)}{1-\cos 4\pi\alpha\,\operatorname{sech}(4\pi\zeta/b)}. \quad . \quad . \quad . \quad . \quad (40)$$

Since $\exp(4\pi\zeta/b)\gg 1$, this may be written, to a good approximation,

$$v=\frac{b^2\mu}{4\pi(1-\sigma)}[1+2\cos 4\pi\alpha\exp(-4\pi\zeta/b)]. \quad . \quad . \quad . \quad (41)$$

On account of the approximate method of calculation employed, the two symmetrical configurations of the dislocation ($\alpha=0$ and $\alpha=\frac{1}{2}$) have the same energy. The argument of the exponential is 2π times the number of atoms in the half width of a dislocation. In a metal, the deviation of the law of force between two planes from the sinusoidal form will in general increase the half width of the dislocation (§ 2.4), and the amplitude of v is reduced. Foreman, Jaswon and Wood have obtained the formula corresponding to (40) for the family of laws of force (16). Their result confirms the exponential decrease in the amplitude of v with increasing width of the dislocation; the amplitude is reduced by the factor $(1+K+\frac{1}{6}K^2)\exp(-K)$, where $K=2\pi(a_0-1)/(1-\sigma)$.

In a lattice such as that of diamond which is held by directed bonds the concept of width of a dislocation loses its meaning, and the approximations on which (40) is based break down completely. It is clear that in such a case the amplitude of v no longer contains this very small exponential factor.

The force acting on unit length of the dislocation is

$$F=-\frac{1}{b}\frac{dv}{d\alpha}=\frac{2b\mu}{1-\sigma}\sin 4\pi\alpha \exp\left[-\frac{4\pi\zeta}{b}\right]. \qquad (42)$$

If the dislocation is to move continuously through the lattice, there must be an external shear stress P producing a force bP exceeding the maximum value of F. This leads to a shear strength

$$P=\frac{2\mu}{1-\sigma}\exp\left[-\frac{4\pi\zeta}{b}\right]. \qquad (43)$$

For a partial dislocation in a close packed lattice, $a=b\sqrt{2}$, and the value of μ/P is given in the following table for various values of σ:

σ	0·2	0·25	0·3	0·35	0·4
μ/P	27,000	52,000	110,000	280,000	810,000

For the simple cubic lattice with $a=b$ and $\sigma=\frac{1}{3}$, μ/P is given as a function of the parameter a_0 in the following table:

a_0	1	2	4	10
μ/P	5000	2×10^6	5×10^{13}	4×10^{35}

Although the simplest assumptions lead to a strength greater than that of a pure single crystal, a flattening of the law of force corresponding to $a_0=3$ is sufficient to make this force negligibly small. Moreover, as Shockley (1947) has pointed out, arguments such as that of the following section show that a dislocation will not usually lie exactly along a line of minimum energy, but will ride across the crests and troughs of the potential v. Then, by arguments similar to those of § 7.3 of Part II, we see that the stress required to move a whole dislocation is much less than the maximum stress acting on any one element of it.

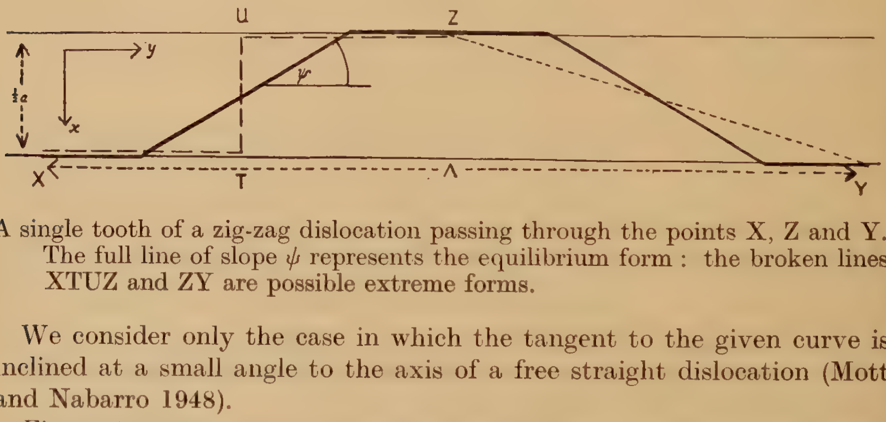
Dietze (1952 a) has suggested that to regard the interaction across the glide plane as acting between lattice points is as crude an approximation as to regard the interaction as acting between continuous surfaces. It is more realistic to consider the centres of force as spreading for some distance around each lattice point; the value of v then lies between the value zero given by the approximation of a continuum and the value (40) given by the approximation of lattice points. Dietze considers in detail several cases in which the spread of a centre of force is given by a Gaussian function $s^{-1}\exp(-x^2/s^2)$, where s is a parameter defining the extent of the spread. For $s=0$ the model reduces to one of lattice points, and for $s=\infty$ to a continuum. A reasonable value of s is probably about $\frac{1}{2}b$. The amplitude of v is reduced by a factor $\exp[-(\pi s/b_0)^2]$, where b_0 is a constant which depends on the nature and orientation of the dislocation. Thermal vibrations are regarded (Dietze 1952 b) as increasing the value of s.

Dietze solves the problem by considering the case of a continuum as a zero order approximation, the difference between the integral equation for the real problem and that for the continuum being regarded as a small perturbation, as is the applied stress P. The condition that the perturbed equation should have a solution is that the perturbation should be orthogonal to the zero order solution, and this condition is expressed as a relation between P and α.

4.2. THE ZIG-ZAG DISLOCATION

In considering a dislocation which approximates to a given curve, we have first to decide whether its equilibrium form is a smooth curve or a series of steps lying parallel to lattice directions and approximating to the smooth curve. We have then to decide by how much the energy of the equilibrium form exceeds that of a straight dislocation.

Fig. 40



A single tooth of a zig-zag dislocation passing through the points X, Z and Y. The full line of slope ψ represents the equilibrium form : the broken lines XTUZ and ZY are possible extreme forms.

We consider only the case in which the tangent to the given curve is inclined at a small angle to the axis of a free straight dislocation (Mott and Nabarro 1948).

Figure 40 represents one complete tooth of a zig-zag dislocation which is constrained to lie along one equilibrium line in the lattice at a series of points such as X and Y, separated by a distance Λ much greater than b, and to lie along the next equilibrium line at a distance $\frac{1}{2}b$ at a series of equidistant points such as Z. We allow it to take any form intermediate between that of the dotted line XTUZ and that of the dashed line ZY. We vary the angle ψ at which the dislocation line crosses from one equilibrium line to the other, and minimize the energy. In practice we carry out the calculation only for small values of ψ, and observe whether the energy increases or decreases as ψ increases from its least possible value, b/Λ.

The energy of the zig-zag dislocation exceeds that of a straight dislocation lying along XY because the energy of misfit in the glide plane is increased, and because the elastic energy of the blocks above and below the glide plane is increased. As ψ increases, the region in which the misfit is unnecessarily bad becomes smaller, and the energy of misfit diminishes, but the same elastic displacements are concentrated into a smaller space, and the elastic energy increases.

We first estimate the change in the energy of misfit. We assume that an element of length dy at a distance x from the line XY has the same energy of misfit as an element of the same length in a dislocation lying parallel to the line XY (the y axis in fig. 33), and passing through the centre of the element. This energy exceeds that of an equal element lying in the line XY by an amount given by (40), and equal to

$$[2\pi(1-\sigma)]^{-1}b^2\mu \exp(-4\pi\zeta/b)(1-\cos 4\pi x/b)\, dy.$$

This expression vanishes over the parts of the zig-zag parallel to XY. The sloping parts of the zig-zag occupy a fraction $b/\Lambda\psi$ of its length, and the average misfit energy per unit length of the zig-zag dislocation exceeds that of a straight dislocation lying along XY by

$$W_m=[2\pi\Lambda\psi(1-\sigma)]^{-1}b^3\mu \exp(-4\pi\zeta/b). \quad . \quad . \quad . \quad . \quad (44)$$

Next we estimate the elastic energy of the zig-zag dislocation. For a straight dislocation along the y axis the displacements u and w are given by (8) of Part II. There is no v component of displacement along the y axis, and u and w are independent of y. We assume that the zig-zag dislocation also produces no v component of displacement, but, near the sloping parts of the zig-zag, u and w now depend on y. The shear strains between neighbouring planes perpendicular to Oy are given by

$$e_{xy}=\tfrac{1}{2}\psi\, \partial u/\partial x, \quad e_{zy}=\tfrac{1}{2}\psi\, \partial w/\partial x. \quad . \quad . \quad . \quad . \quad . \quad (45)$$

The strain energy per unit length near the sloping parts exceeds that of a straight dislocation by

$$2\mu\iint (e_{xy}{}^2+e_{zy}{}^2)\, dx\, dz=\tfrac{1}{2}\mu\psi^2\iint\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right] dx\, dz. \quad . \quad . \quad (46)$$

(This corrects a numerical error in the original paper.) If this integral were extended to infinite values of x and z it would diverge logarithmically. It is, however, clear that our approximation of considering that the displacement of atoms in any plane perpendicular to Oy depends only on the point at which the dislocation meets that plane, and not at all on the displacements of atoms in neighbouring planes, must break down in regions of the plane distant Λ or more from the dislocation. The variations from plane to plane become smaller at these distances, and the shear strain energy (46) is reduced. An estimate of the rate at which the shear energy falls off with distance is given in the Appendix to the original paper. The result is to multiply the integrand in (46) by a factor $\exp[-4\pi(x^2+z^2)^{1/2}/\Lambda]$. This new integral is taken over a fraction $b/\Lambda\psi$ of the total length of the dislocation, and the average elastic energy per unit length of the zig-zag dislocation exceeds that of the straight dislocation by

$$W_e=(\mu b\psi/\Lambda)I(\Lambda/b),$$

where

$$I(\Lambda/b)=\iint\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right]\exp\left[-\frac{4\pi(x^2+y^2)^{1/2}}{\Lambda}\right]dx\,dz. \quad . \quad . \quad (47)$$

The values of $\partial u/\partial x$ and $\partial w/\partial x$ must be inserted from § 2.1 of Part II. The approximate value of the integral $I(\Lambda/b)$ is derived in the paper quoted, and we finally obtain

$$W_e=\frac{kb^3\mu\psi}{16\pi\Lambda}\ln\frac{\Lambda}{4\pi\zeta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

where $(3-2\sigma)^2>k>(1-2\sigma)^2$, and $\zeta=a/2(1-\sigma)$. Here the numerical factor 1.3 used in the original derivation has been replaced by the more accurate value $\frac{1}{2}\pi$. We shall take $k=1$ as an estimate of the true value.

The total energy per unit length is

$$W=W_m+W_e=\frac{b^3\mu}{2\pi\Lambda}\left[\frac{\exp(-4\pi\zeta/b)}{1-\sigma}\frac{1}{\psi}+\frac{\ln(\Lambda/4\pi\zeta)}{8}\psi\right]. \quad . \quad (49)$$

Varying ψ, we obtain a minimum of W when

$$\psi=\left[\frac{8\exp(-4\pi\zeta/b)}{(1-\sigma)\ln(\Lambda/4\pi\zeta)}\right]^{1/2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

To obtain a numerical value, we take $\sigma=\frac{1}{3}$, $a=\sqrt{2}b$, $\Lambda=100b$, giving

$$\psi=2{\cdot}44\exp(-3\pi/\sqrt{2})=3{\cdot}1\times10^{-3}.$$

This value of ψ is small, so that our original assumption is self-consistent. However, the minimum value of ψ allowed by the geometry of fig. 40 is $\psi=b/\Lambda=10^{-2}$, and this value, which is also small, is the angle giving the lowest possible energy. If Λ is rather larger, ψ takes the value given by (50). The energy is then given by

$$W=\frac{b^3\mu}{\pi\Lambda}\left[\frac{\ln(\Lambda/4\pi\zeta)}{8(1-\sigma)}\right]^{1/2}\exp(-2\pi\zeta/b), \quad . \quad . \quad . \quad (51)$$

which, with the numerical values taken above, is

$$W=0{\cdot}61(b^3\mu/\pi\Lambda)\exp(-2\pi\zeta/b). \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

A straight dislocation of length Λ has become a zig-zag of length $\Lambda+b(\operatorname{cosec}\psi-\cot\psi)\simeq\Lambda+\frac{1}{2}b\psi$, and its energy has increased by ΛW. The tension E in it is, therefore, $2\Lambda W/b\psi$, which is

$$E=\frac{b^2\mu}{4\pi}\ln\frac{\Lambda}{4\pi\zeta}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

With the same numerical values, this gives

$$E=0{\cdot}16\,b^2\mu. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

We conclude that a dislocation in a strained lattice will approximate to a smooth curve rather than to a square zig-zag, and that its effective tension when it deviates little from a straight line is approximately $0{\cdot}16\, b^2\mu$. This value may be compared with that given by (119) of Part II for a circular dislocation, in which a logarithmic term also appears, but the numerical factor $1/4\pi$ is replaced by $5/16\pi$.

If ψ is given by $\psi = b/\Lambda$ instead of by (50), the second term in (49) is larger than the first, and the energy per unit length is

$$W = \frac{\theta b^4 \mu}{16\pi\Lambda^2} \ln \frac{\Lambda}{4\pi\zeta},$$

where θ lies between 1 and 2. The increased length of the dislocation is now $b^2/2\Lambda$, and E is given by

$$E = \frac{\theta b^2 \mu}{8\pi} \ln \frac{\Lambda}{4\pi\zeta},$$

which lies between the value given by (133) and half of that value. The general conclusions are unaltered.

We may obtain (48) by an alternative method. Two segments of the dislocation were originally of total length b/ψ and of pure edge type. The original energy per unit length was approximately

$$\frac{b^2\mu}{4\pi(1-\sigma)} \ln \frac{\Lambda}{4\pi\zeta},$$

so that the original energy of these two segments was

$$\frac{b^3\mu}{4\pi(1-\sigma)\psi} \ln \frac{\Lambda}{4\pi\zeta}.$$

The segments are now inclined at an angle ψ to their original direction, while their Burgers vector is unaltered. They are now partly of screw type, and their energy per unit length is reduced to

$$\left(\frac{b^2\mu}{4\pi} \ln \frac{\Lambda}{4\pi\zeta}\right)\left(\frac{\cos^2\psi}{1-\sigma} + \sin^2\psi\right) = \left(\frac{b^2\mu}{4\pi(1-\sigma)} \ln \frac{\Lambda}{4\pi\zeta}\right)(1-\sigma\psi^2).$$

The length has increased from b/ψ to $b \sec \psi/\psi$, and the total energy is now

$$\left(\frac{b^3\mu}{4\pi(1-\sigma)\psi} \ln \frac{\Lambda}{4\pi\zeta}\right)(1-\sigma\psi^2)(1+\tfrac{1}{2}\psi^2).$$

Since σ is necessarily less than $\frac{1}{2}$, the increased length outweighs the reduction in the energy of unit length, and we obtain

$$W_e = \frac{(1-2\sigma)b^3\mu\psi}{8\pi(1-\sigma)\Lambda} \ln \frac{\Lambda}{4\pi\zeta},$$

which agrees with (48) if we take $k = 2(1-2\sigma)/(1-\sigma)$. For $\sigma = \frac{1}{3}$, this coincides with the value $k = 1$ already assumed.

The value $E = 0{\cdot}16\, b^2\mu$ obtained in (54) is smaller by a factor of about 5 than the estimate of the line energy of a dislocation obtained by other methods (Frank 1950 a). This result arises partly from the small value of Λ, here assumed to be $100\,b$, a value probably appropriate to a precipitation-hardened alloy. For a soft material, the value $\Lambda = 1\,000\,b$ would be more reasonable, which increases the value of E to $0{\cdot}34\,b^2\mu$. The rest of the discrepancy corresponds to the appearance of a numerical factor $1/4\pi$ in the logarithm in (53), whereas the corresponding factor in (119) of Part II is $2/e$. Clearly the approximations used in deriving (54) could introduce errors greater than this. However, the difference may be real, for there is no obvious reason why the energy per unit length required to bend a straight dislocation into a sinusoidal curve of small amplitude and wavelength Λ should be precisely equal to the total energy per unit length of a circular dislocation of radius Λ, or to that of a straight dislocation lying along the axis of a cylinder of radius Λ.

§ 5. DISLOCATIONS IN SPECIAL CRYSTAL STRUCTURES

So far, we have dealt with those properties of dislocations which are common to all crystals, both those which are lattices and those which have more than one atom in the unit cell. We now turn to those properties which distinguish one crystal structure from another, and examine such problems as ' What determines the glide system in any particular crystal ? Why do metals with cubic close packing work harden more rapidly than those with hexagonal close packing ? Why does the body-centred cube give deformation twins, whereas the face-centred cube does not ? '. The answers to these questions usually depend on the idea of partial dislocations. Other new ideas, in particular those of dislocation reactions and dislocation nodes, can be illustrated from the properties of complete dislocations. Throughout the discussion we shall draw freely on the papers of Frank (1951 a) and Read and Shockley (1952).

5.1. REACTIONS AND NODES

Two dislocations with Burgers vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ may combine to form a single dislocation with Burgers vector $\mathbf{b}_3 = \mathbf{b}_1 + \mathbf{b}_2$. This is a *dislocation reaction*. We may illustrate it by considering the dislocations $[\tfrac{1}{2}a, \tfrac{1}{2}a, 0]$ and $[-\tfrac{1}{2}a, 0, \tfrac{1}{2}a]$, which both lie in the $(1\bar{1}1)$ glide plane of a face-centred cubic lattice of parameter a. Suppose these dislocations both lie along lines parallel to $[21\bar{1}]$. Then (cf. § 4.63 of Part II) they attract one another, and coalesce. By considering the residue of displacement for a circuit surrounding both dislocations, which is unaltered by the motion and coalescence of the two dislocations, we see that the resulting dislocation has Burgers vector $[0, \tfrac{1}{2}a, \tfrac{1}{2}a]$. This reaction is represented by the equation

$$\tfrac{1}{2}a[110] + \tfrac{1}{2}a[\bar{1}01] = \tfrac{1}{2}a[011]. \quad . \quad . \quad . \quad . \quad . \quad (55)$$

This dislocation, which was made in the plane $(1\bar{1}1)$, also lies in the close packed plane $(11\bar{1})$. It can dissociate in this plane according to the scheme

$$\tfrac{1}{2}a[011] = \tfrac{1}{2}a[\bar{1}10] + \tfrac{1}{2}a[101]. \quad . \quad . \quad . \quad . \quad . \quad (56)$$

The reactions (55) and (56) differ in that the former proceeds with release of energy, whereas the latter absorbs energy. If we neglect the factor $(1-\sigma)$ which relates the formulae for the contributions of the edge and screw components to the total energy of a dislocation, we have a simple rule for deciding whether a reaction of the type $\mathbf{b}_1+\mathbf{b}_2\rightarrow\mathbf{b}_3$ releases energy. The initial energy is proportional to $b_1^2+b_2^2$, the final energy to $(\mathbf{b}_1+\mathbf{b}_2)^2=b_1^2+b_2^2+2\mathbf{b}_1\mathbf{b}_2$. Thus energy is released if $\mathbf{b}_1\mathbf{b}_2$ is negative, that is, if the angle between $\mathbf{b}_1$ and $\mathbf{b}_2$ is greater than a right angle. The neglect of the factor $(1-\sigma)$ is usually justified because we are already neglecting the influence of the crystal anisotropy on the energy of a dislocation.

In general, a dislocation which is not the shortest possible in a lattice will dissociate. Thus, in the face-centred cubic lattice, the double dislocation $a[110]$ dissociates into two dislocations $\frac{1}{2}a[110]$. The dislocation reaction $a[100]\rightarrow\frac{1}{2}a[110]+\frac{1}{2}a[1\bar{1}0]$ involves no change of energy in the present approximation. However, the dislocations such as $\frac{1}{2}a[110]$ can reduce the total energy by extending in the close packed planes (§ 5.2), while the dislocation $a[100]$ cannot. In the body-centred cubic lattice, the reaction $a[100]\rightarrow\frac{1}{2}a[111]+\frac{1}{2}a[1\bar{1}\bar{1}]$ increases the energy, since the angle between $[111]$ and $[1\bar{1}\bar{1}]$ is obtuse. The dislocation $a[100]$, though not the shortest possible, is stable. Such dislocations may be removed by reactions of the type

$$a[100]+a[010]\rightarrow\tfrac{1}{2}a[111]+\tfrac{1}{2}a[11\bar{1}], \quad . \quad . \quad . \quad . \quad (57)$$

involving four dislocations.

Two dislocations which lie in the same plane may combine to form a third even though their lines are not parallel, provided their lines and Burgers vectors all lie on the same plane. The dislocations $\mathbf{b}_1$ $\mathbf{b}_2$ $\mathbf{b}_3$ then meet to form a *node*. Frank has suggested that it is then convenient to use a new sign convention, the Burgers circuit for each dislocation being taken in such a direction that it appears to be right handed when looking outwards from the node. The three dislocations are treated equally, and their Burgers vectors satisfy the symmetrical relation $\mathbf{b}_1+\mathbf{b}_2+\mathbf{b}_3=0$. More than three dislocations may meet at a node. If we remove the restriction that a reaction such as $\mathbf{b}_1+\mathbf{b}_2\rightarrow-\mathbf{b}_3$ can occur by a gliding of the dislocations which moves the node, we no longer require that the three lines and the three Burgers vectors should lie in one plane. Thus the four dislocations in (57) can form a node, but the reaction (57) cannot occur by glide. Three dislocations free to move in the same glide plane can only form a stable node if the angles between their Burgers vectors are all obtuse, so that the triangle with sides $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ is acute angled. This is because the equilibrium of the node requires that the tensions in the dislocations should form a triangle of forces with sides proportional to b_1^2, b_2^2 and b_3^2. Hence $b_1^2<b_2^2+b_3^2$, and $\mathbf{b}_2\,\mathbf{b}_3<0$. This is just the condition that the energy should fall when $\mathbf{b}_2$ and $\mathbf{b}_3$ combine to give $-\mathbf{b}_3$. Similarly $\mathbf{b}_3\,\mathbf{b}_1<0$ and $\mathbf{b}_1\,\mathbf{b}_2<0$.

5.2. PARTIAL DISLOCATIONS

If a crystal forming a lattice is cut along a half plane lying between two sheets of atoms, and the cut surfaces are displaced relative to each other by a lattice vector lying in the plane of the cut, we say that the crystal has slipped on this plane, and the boundary between the slipped and the unslipped regions is called a *complete dislocation*. If the Burgers vector of the relative displacement does not lie in the plane of the cut, one or more layers of atoms must be inserted between the cut surfaces or removed from them. If the displacement is still a lattice vector, the crystal is still perfect except near the edge of the region inserted or removed, which is again a complete dislocation. If the crystal structure has a basis, the same process may be carried out: the regions added or removed must be made of *complete* cells.

It often happens in real crystals that a relative displacement of the two cut surfaces by an amount which is not a lattice vector, possibly combined with the addition or removal of layers of atoms, produces a mechanically stable configuration.

The configurations assumed to be of low energy are usually those which are observed to occur in the composition planes of twinned crystals. The region of the cut is now a fault plane, and its edge is a *partial dislocation*. The properties of partial dislocations with their Burgers vectors in the fault plane are entirely different from those of other partial dislocations; we discuss the two types in § 5.21 and § 5.22. The formal definition of the Burgers vector of a partial dislocation presents certain difficulties, particularly in the case in which a layer of atoms has been inserted which does not belong to the structure of the original crystal above or below the cut. It seems necessary for the Burgers circuit to begin and finish on the cut, in a region free from large elastic strains.

5.21. *Mobile Partial Dislocations*

If the Burgers vector of a partial dislocation lies in the fault plane, it can glide in this plane just as a complete dislocation glides in the slip plane. In the absence of any applied stress, there is a force on unit length of the partial dislocation tending to move it in the direction which will reduce the area of the fault. This force is equal to the surface energy γ of the fault plane. For a stacking fault in copper, $\gamma = 18$ dyne cm^{-1} (Fisher and Dunn 1952). If the partial dislocation moves out of the fault plane, it leaves behind it a region of misfit which in general has a very high energy, probably at least as large as the energy of a large angle grain boundary, which in copper is 535 dyne cm^{-1}.

The most important partial dislocations of this kind are those occurring in the close packed metals. The cubic and hexagonal close packed structures both contain close packed sheets of atoms, and the relation between one sheet and the next is the same in both structures. They can only be distinguished by considering at least three planes. In the cubic lattice these lie (in the usual notation) A B C, in the hexagonal, ABA. If only forces between nearest neighbours were important, there would occur not only the regular structures ABCABCABCABC . . . and . . . ABABABABABAB . . . and the cubic twin . . . ABCABCBACBAC . . . , but also the faulted hexagonal structure . . . ABABABCBCBCB . . . and such irregular structures as . . . ABCBCBCBCABC . . . , which contain layers of cubic and hexagonal packing. Any arrangement in which the same letter does not occur twice consecutively is permissible. In practice, many metals change from cubic to hexagonal close packing with change of temperature, while cobalt in particular shows irregular piling. It therefore seems that the crystal structure of metals is determined largely by nearest neighbour repulsions, and that the model of a pile of hard spheres provides a useful analogy. It is at once evident on handling such a model (Millington and Thompson 1924, Mathewson 1944) that the easiest movement is not in the glide direction $[1\bar{1}0]$ but in the twinning direction $[1\bar{2}1]$, where the plane is taken to be a cubic (111) plane. This leads from the cubic lattice . . . ABCABCABCABC . . . to the arrangement . . . ABCABCBCABCA . . . , or from the hexagonal structure . . . ABABABABABAB . . . to the arrangement . . . ABABABCACACA . . . . If this movement occurs over part of a plane, the boundary of the slipped region is an imperfect dislocation with Burgers vector $\frac{1}{6}a[1\bar{2}1]$. Partial dislocations of this kind were first discussed by Heidenreich and Shockley (1948). Two such partial dislocations may combine to give a complete dislocation according to the scheme

$$\tfrac{1}{6}a[1\bar{2}1]+\tfrac{1}{6}a[2\bar{1}\bar{1}]=\tfrac{1}{2}a[1\bar{1}0]. \quad . \quad . \quad . \quad . \quad . \quad (58)$$

The resolution of a dislocation $\frac{1}{2}a[1\bar{1}0]$ into two dislocations $\frac{1}{6}a[1\bar{2}1]$ and $\frac{1}{6}a[2\bar{1}\bar{1}]$, all lying in the (111) plane, is illustrated in Plate IX, where the plane of the figure is taken as (111) and the horizontal direction as $[1\bar{1}0]$. The two partial dislocations are readily seen by looking along $[10\bar{1}]$ and $[01\bar{1}]$ directions; the existence of a dislocation $\frac{1}{2}a[1\bar{1}0]$ is most easily verified by counting steps round the circuit indicated. However, there is no exact analogy, for the glide plane in Plate IX is not the (111) plane of the figure, but the plane $(11\bar{1})$. If the extra rows of atoms along $[10\bar{1}]$ and $[01\bar{1}]$ are continued as extra planes $(1\bar{1}1)$ and $(\bar{1}11)$ respectively, the line along which they meet is [110], which does not lie in the glide plane. The two partial dislocations of Plate IX cannot separate, but since the angle between them is acute, the complete dislocation in the real crystal would always dissociate if the irregular piling were not a state of higher energy than the regular. When the partials are separated by a distance r, the force of repulsion on unit length of each is of order $\mathbf{b}_1\,\mathbf{b}_2\mu/2\pi r=a^2\mu/24\pi r$. If the energy of unit area of the surface of misfit is γ, the force on unit length tending to bring the partials together again is γ. Their equilibrium separation is thus given by

$$r=a^2\mu/24\pi\gamma.$$

Heidenreich and Shockley suggest that γ may be estimated from the energy change E per atom in the transition from hexagonal to cubic close packing according to a formula of the type $\gamma=8E/\sqrt{3}a^2$, which gives $r=\sqrt{3}a^4\mu/192\pi E$, and should be of the right order of magnitude. They suggest that E may be about 100 cal $\text{mol}^{-1}=7\times10^{-15}$ erg per atom, while $a^3\mu\simeq5\times10^{-11}$ erg per atom, giving a separation of $r=20a$. This suggests that cases may occur in practice for which the two partial dislocations are tightly bound together, and other cases in which their binding is so weak that quite small stresses will separate them. The approximate treatment of Frank and van der Merwe (1949 c) (§ 3.1) may be applied to this problem. Leibfried and Dietze (1951) have attempted to find solutions using an extension of Peierls's method, allowing for the two components of the displacement in the glide plane. They find approximate solutions by a variational method, minimizing the energy. Results are given for the (100) plane, and also for half dislocations lying along various lines in the (111) plane. A solution is also given for a pair of partial dislocations which combine to form a complete dislocation in the (111) plane. Since their approximation neglects the energy γ of a stacking fault, there should be no equilibrium until the partial dislocations have repelled each other to infinity. In fact, they find a solution in which the partial dislocations are separated by only a few interatomic spacings. The erroneous result arises because the form of solution they assume, while adequate for very small separations of the partial dislocations, cannot represent two partial dislocations widely separated; for wide separations it represents two dislocations with Burgers vectors nearly antiparallel.

Suppose the dislocation $\frac{1}{2}a[1\bar{1}0]$ lies along the line $[11\bar{2}]$, so that it is of pure edge type. The partial dislocations have equal edge components $\frac{1}{4}a[1\bar{1}0]$, and opposite screw components $\frac{1}{12}a[\bar{1}\bar{1}2]$ and $\frac{1}{12}a[11\bar{2}]$. A shear stress P acting across the glide plane in the glide direction $[1\bar{1}0]$ drives them both in this direction. A shear stress P acting in the direction $[11\bar{2}]$ acts on the screw components, driving them apart with a force per unit length of $Pa/2\sqrt{6}$. They will separate completely under a stress given by $P=2\sqrt{6}\gamma/a$.

Dislocations in which the two partials are appreciably separated are referred to by Heidenreich and Shockley as *extended dislocations*. They do not have the same freedom of motion as complete dislocations, because the two partials have Burgers vectors which are not parallel, and there is no line along which they are simultaneously pure screws. Since only a pure screw can move out of the glide plane, there is an activation energy required in order to produce the reaction

$$\tfrac{1}{6}a[121]+\tfrac{1}{6}a[2\bar{1}\bar{1}]=\tfrac{1}{2}a[110]=\tfrac{1}{6}a[12\bar{1}]+\tfrac{1}{6}a[211], \quad . \quad . \quad (59)$$

which leads to a dislocation extended in the plane $(11\bar{1})$.

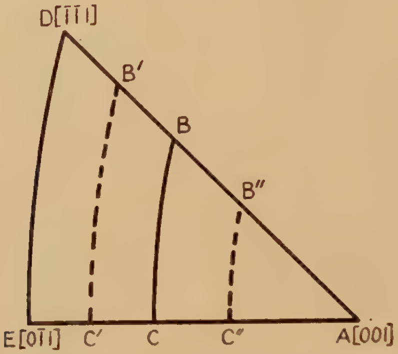
Thompson (1952) has considered a dislocation $\frac{1}{2}a[1\bar{1}0]$ which already lies partly in the plane (111) and partly in (111). The partial dislocation

$\frac{1}{6}a[1\bar{2}1]$ in (111) may continue as either of the partials $\frac{1}{6}a[1\bar{2}\bar{1}]$ or $\frac{1}{6}a[2\bar{1}1]$ in $(11\bar{1})$. In the first case it is linked to the join of the other pair of partials by a dislocation $\frac{1}{3}a[001]$, and in the second case by a dislocation $\frac{1}{6}a[\bar{1}\bar{1}0]$. The pair of partial dislocations in (111) is separated by a stacking fault and the pair in $(11\bar{1})$ by a different stacking fault. The small dislocations $\frac{1}{3}a[001]$ and $\frac{1}{6}a[\bar{1}\bar{1}0]$ lie along the junction of these two stacking faults. Thompson has also considered the behaviour of the stacking faults at a node corresponding to eqn. (55).

If the partial dislocations travel independently through the lattice, leaving surfaces of misfit behind them, their motion is more constrained than that of a complete dislocation. A complete dislocation may be reflected at a free boundary with change of sign, or it may be followed across the glide plane by another similar dislocation. In either case the resulting displacement is $2\mathbf{b}$. If $\mathbf{b}$ is a partial dislocation, the displacement $2\mathbf{b}$ does not lead to close packing, but to an unstable position of high energy. It is therefore necessary for the second partial dislocation to have its Burgers vector inclined to that of the first. The resultant of the two then lies along a $<110>$ slip direction. Alternatively, the same displacement $\mathbf{b}$ may occur in successive glide planes of the cubic lattice, leading to the formation of a twinned lamella. In the hexagonal structure a displacement $\mathbf{b}$ which is produced between layers 1 and 2 by the passage of a partial dislocation cannot occur between layers 2 and 3. If the three possible partial dislocations between the first two layers are $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$, those possible between the next two layers are $-\mathbf{b}_1$, $-\mathbf{b}_2$ and $-\mathbf{b}_3$. The partial dislocation $\mathbf{b}_1$ can travel only on alternate glide planes, and in doing so it creates a lamella of cubic crystal.

Mathewson (1951) has considered the effect of partial dislocations on the law of resolved shear stress. He did not use the terminology of dislocation theory, but we present his argument in this form. Figure 41 represents a unit triangle in the stereographic projection. According to the analysis of Taylor and Elam (1923), a crystal deformed in tension will slip in a particular $<110>$ direction on a particular $\{111\}$ plane if the tensile axis lies within this triangle. If slip along $[\bar{1}01]$ consists of successive movements along $[\bar{2}11]$ and $[\bar{1}12]$, the triangle ADE must be divided by a line BC into two regions, so that in ABC the resolved shear stress is greater in the $[\bar{1}12]$ direction than in the $[\bar{2}\bar{1}1]$ direction, while in BCED this relation is reversed. If a tension is applied with an axis in the area ABC, and gradually increased, the first effect is to move partial dislocations $\frac{1}{6}a[\bar{1}12]$ across their glide plane $(1\bar{1}1)$. Slip cannot continue unless the partial dislocations $\frac{1}{6}a[\bar{2}\bar{1}1]$ then follow them across the glide plane. If we make no allowance for the energy γ of a stacking fault, this means that a tension applied in the triangle ABC will cause slip only when its component resolved into the $(1\bar{1}1)$ plane and the $[\bar{2}\bar{1}1]$ direction reaches a critical value. It is usually assumed to be the component in the direction $[\bar{1}01]$ which is constant; if, in fact, the component in the $[\bar{2}\bar{1}1]$ direction is constant, the $[\bar{1}01]$ component varies by about $\frac{1}{3}$ of its value as the tensile axis moves through the triangle. The observed variation (Fenn, Hibbard and Lepper 1950) in 70–30α brass was only 1/27 of the mean value over a wide range of orientations. If we take into account the energy γ of the stacking fault, the resolved shear stress required to heal a fault is less than that to create it. There must be some other locus B′C′ such that a tension along a direction lying in B′C′ which has a component resolved along $[\bar{2}\bar{1}1]$ just large enough to create a stacking fault has also a (smaller) component resolved along $[\bar{1}12]$ just large enough to heal this stacking fault. We may now extend Mathewson's argument, and introduce the locus B″C″ of directions of tension for which the making of a fault with a $[\bar{1}12]$ partial dislocation and its healing with a $[\bar{2}\bar{1}1]$ partial dislocation require the same tension. In AB″C″, the partial dislocations are dissociated by the stress, and it is the stress resolved in the $]\bar{2}\bar{1}1]$ direction which determines the strength.

Fig. 41



A unit triangle of the stereographic projection. If the crystal is extended along an axis lying in the triangle ADE, slip occurs on one particular glide system; outside this triangle other systems are preferred. In ABC ‘easy glide’ along $[\bar{1}12]$ occurs before ‘easy glide’ along $[\bar{2}\bar{1}1]$. In AB″C″ the applied stress required to create a stacking fault by this process is less than that required to heal an existing stacking fault by glide along $[\bar{2}\bar{1}1]$.

Similarly, in B′C′ED, it is the $[\bar{1}12]$ component which must reach a critical value. In B′B″C″C′, the partial which moves first is immediately followed by the other partial, healing the fault. The effective stress on the first partial is not only the applied stress, but also its interaction with the second partial, which pushes it further ahead. The partials are not dissociated, but travel together across the glide plane under the influence of the [101] component of the stress. Apparently this region B′B″C″C′ covers most or all of the unit triangle in the case of α brass. Since this is a material which readily twins and so presumably has a rather low value of γ, this will probably occur in most other face-centred cubic metals.

5.22. *Sessile Dislocations*

The idea of a *sessile* dislocation, which cannot glide in the lattice, was introduced by Frank (1949 c). Any partial dislocation with its Burgers vector inclined to the plane of the stacking fault is sessile; other dislocations, which can move by glide under moderate stresses, have been called *glissile*. We first discuss the type of sessile dislocation which Frank described.

If Plate IX represents the (111) plane of a close packed cubic lattice, the dislocation shown may be regarded as having been produced by inserting two close packed half planes of atoms into the crystal, lying parallel to $(1\bar{1}1)$ and $(\bar{1}11)$ respectively. The partial dislocation formed by inserting one of these half planes alone is perhaps the simplest imperfection of dislocation type that can exist in a close packed cubic lattice. If, for example, the crystal is considered to be made by piling (111) planes, a plane of type C may be inserted between any two consecutive planes A and B without spoiling the close packing. Similarly, a portion of a close packed plane may be removed. In the hexagonal packing, a portion of a faulted plane may be added, but the removal of a portion of a plane would spoil the close packing. The Burgers vector of this partial dislocation is $\frac{1}{3}a[111]$, and it can only glide in a plane containing this direction. The dislocation only leaves a close packed lattice if it moves in a (111) plane, and this does not contain the Burgers vector. It is therefore locked in its position in the lattice. Partial dislocations of this kind can move by three processes: (*a*) the diffusion of individual atoms, (*b*) dissociation into mobile dislocations, (*c*) ineffective glides. Process (*a*) is slow. Process (*b*) can occur by a reaction such as

$$\tfrac{1}{3}a[111]\rightarrow\tfrac{1}{2}a[011]+\tfrac{1}{6}a[2\bar{1}\bar{1}], \qquad \ldots\ldots \quad (60)$$

where all three dislocations lie along the line $[\bar{1}10]$ which is the intersection of the plane (111) in which the first dislocation can lie and the third can glide with the plane $(11\bar{1})$ in which the second can glide. The energy increases from $\frac{1}{3}$ to $\frac{2}{3}$ in this process. On the other hand, if a wandering partial dislocation $\frac{1}{6}a[\bar{2}11]$ meets the sessible dislocation, they can combine without change of energy to form a mobile complete dislocation. As an example of (*c*) we consider the result of removing part of a C sheet from a cubic lattice, giving . . . A B C A B C A B A* B C A B C . . . The plane marked with an asterisk can move between the B planes to a C position, its movement with respect to each being in a direction of easy glide. The resulting structure . . . A B C A B C A B C B* C A B C . . . can move by a similar ineffective glide to . . . A B C A B C A B C A C* A B C . . . , and so on.

It seems certain that sessile dislocations will play an important part in the process of work hardening, because they form fixed sources of internal stress which interact with mobile dislocations. In addition to the dislocation $\frac{1}{3}a[111]$, the face-centred cubic lattice can contain the sessile dislocations $\frac{1}{3}a[001]$ and $\frac{1}{6}a[110]$ which arise at the junctions of stacking faults on different $\{111\}$ planes. These last two dislocations do not exist in the hexagonal structure. The first can exist, but cannot be formed by the interaction of mobile dislocations, whereas in the cubic lattice the reverse reaction to (60) allows two mobile dislocations to combine, forming a sessile dislocation and releasing energy. If there is only one glide plane, any dislocations which come together by glide have their Burgers vectors in this plane. The results of their reaction also have Burgers vectors in the glide plane, and are not sessile. This explains the slow work hardening of hexagonal metals.

All the partial dislocations of the close packed cubic lattice may be expressed as combinations of partial dislocations of the type $\frac{1}{6}a[211]$. We have already outlined the way in which Frank's sessile dislocation can be formed from one of these and a complete dislocation. Cottrell (1952 c) has developed an idea of Lomer (1951) to show how a sessile partial dislocation $\frac{1}{6}a[110]$ may be formed from complete dislocations alone. Lomer considers a dislocation $\frac{1}{2}a[10\bar{1}]$ lying in (111) and a dislocation $\frac{1}{2}a[011]$ lying in $(11\bar{1})$. These meet along the line $[1\bar{1}0]$, combining to form a dislocation $\frac{1}{2}a[110]$, with release of energy. The line and the Burgers vector define the slip plane as (001), which is not a usual slip plane for the face-centred lattice. The dislocation cannot extend in this plane, and therefore the resistance to its motion is likely to be larger than the resistance which a dislocation meets if it lies in $\{111\}$. However, it is not sessile. Cottrell points out that it can dissociate according to the scheme

$$\tfrac{1}{2}a[110] \rightarrow \tfrac{1}{6}a[11\bar{2}] + \tfrac{1}{6}a[112] + \tfrac{1}{6}a[110], \quad . \quad . \quad . \quad . \quad (61)$$

the energy falling from $\frac{1}{2}$ to 7/18. The first two products move away in the planes (111) and $(11\bar{1})$ respectively. They leave stacking faults behind them, and are therefore not repelled to infinity, but only to distances of the order of $10a$. The resulting group of three dislocations cannot glide.

It is perhaps more accurate to imagine the initial dislocations as already extended in the planes (111) and (111). In this case the two partial dislocations $\frac{1}{6}a[2\bar{1}\bar{1}]$ and $\frac{1}{6}a[121]$, one in each plane, combine to give $\frac{1}{6}a[110]$, while the other two partial dislocations are unaltered by the reaction.

The principal difficulty in relating this reaction to the experimental observations is that the sessile dislocations are formed on the line $[1\bar{1}0]$, which makes an angle of 60° with the slip directions [101] and [011], whereas the observed deformation bands are planes perpendicular to the active slip direction.

5.3. THE CHOICE OF A GLIDE SYSTEM

The passage of a dislocation of Burgers vector $\mathbf{b}$ translates part of the crystal a distance $\mathbf{b}$ with respect to the rest, and the resulting crystal is again perfect. It follows that $\mathbf{b}$ is a lattice vector joining equivalent atoms in the crystal. The energy of the dislocation is proportional to b^2, and we expect that the glide directions which will be active in practice will be those corresponding to the smallest permissible values of b^2 (Frank 1949 d). This is not because we expect a Maxwell–Boltzmann distribution of dislocations in thermal equilibrium, but because dislocations of large Burgers vector are mechanically unstable. Thus in a cubic lattice a dislocation $2a[100]$, with energy 4, dissociates into two dislocations $a[100]$ widely separated, with total energy 2. The ease with which a dislocation can move through the lattice is probably also influenced by the value of b. A given applied stress exerts on the dislocation a force proportional to b. Some of the forces resisting its motion are also proportional to b, but others depend on higher powers of b, and these make the mobility a decreasing function of b.

These considerations determine the glide *direction*. The choice of glide *plane* is not so clear, which agrees with the observation (Andrade and Chow 1940) that the glide direction is usually a fixed characteristic of the crystal class, while the glide plane may be a function of temperature. Chalmers and Martius (1951) have shown that the glide plane is always the most closely packed plane which contains the Burgers vector; if several planes are observed, these are the most closely packed, the next most closely, and so on. They explain this by assuming that if the spacing of the glide planes is a, while the Burgers vector is $\mathbf{b}$, the energy is an increasing function of b/a. It follows from (6) and (7) that on Peierls's approximation this argument is not correct. The energy is proportional to b^2, but depends on a only through the logarithmic term in ζ. The change of energy for a moderate change in a is likely to be small compared with the change produced by the elastic anisotropy when the glide plane is rotated round the Burgers vector, and this latter effect has been neglected. Probably the most important effect is the deviation of the law of force between planes of atoms from the sinusoidal form. This allows the dislocation to lengthen and so reduce its energy and the resistance to its motion if it lies in certain planes. This is particularly effective in close packed planes, where the dislocation can extend into two partial dislocations.

5.31. *Special Lattices*

5.311. *The Simple Cubic Lattice.*—In the simple cubic lattice of parameter a, all atoms are equivalent. The shortest possible lattice vectors are of the forms $a[100]$ and $a[110]$. If the dislocation $a[110]$ dissociates according to the scheme:

$$a[110] \to a[100] + a[010], \qquad \ldots \ldots \ldots \quad (62)$$

the strain energy is unaltered, and more delicate considerations would be required to decide whether the dislocations present after annealing are $<110>$ or $<100>$. However, since the dissociation (62) requires no energy, a stress acting on a dislocation $a[110]$ which produces a larger force on the component $a[100]$ than on the component $a[010]$ is able to move the component $a[100]$ while the other is still fixed in the lattice.

The glide direction is thus necessarily $<100>$. There are three basic slip vectors $a[100]$, $a[010]$ and $a[001]$ which are mutually perpendicular; nodes and reactions such as (55) are impossible. Salts having the caesium chloride structure are crystallographically simple cubic. Thallium halides, which have this structure, do in fact show $<100>$ as the glide direction, with $\{110\}$ as the glide plane (Smakula and Klein 1949). These salts cannot contain partial dislocations of low energy because of the electrostatic repulsion of like ions. No dislocation reactions of any kind are possible, and a dislocation moving in a particular glide plane can only continue to move in this plane. This is probably the explanation (Nabarro 1950) of the 'prismatic punching' observed in these salts: an indentation on one face of a crystal travels through the crystal and produces mounds on the other faces at points lying in $<100>$ directions from the indentation. (Seitz (1950 c) has given a different explanation. It could also be an example of kinking (Jillson 1950).)

5.312. *The Face-centred Cubic Lattice.*—The shortest lattice vector is $\frac{1}{2}a[110]$, and the glide direction is $<110>$. In the approximation of an isotropic elastic continuum there is no change of energy if the dislocation $a[100]$ dissociates into $\frac{1}{2}a[110]+\frac{1}{2}a[1\bar{1}0]$ or into $\frac{1}{2}a[101]+\frac{1}{2}a[10\bar{1}]$. However, the last four dislocations lie in $\{111\}$ planes in which they can extend and reduce their energy, whereas $a[100]$ does not lie in this plane. Thus the glide plane $\{111\}$ and glide direction $<110>$ are determined.

5.313. *The Body-centred Cubic Lattice.*—The shortest lattice vectors are $a[100]$ and $\frac{1}{2}a[111]$, with energies 1 and $\frac{3}{4}$ respectively. The glide direction is thus $<111>$, and the glide plane is variable. A dislocation $a[100]$ cannot dissociate spontaneously into $\frac{1}{2}a[111]+\frac{1}{2}a[1\bar{1}\bar{1}]$, because the energy rises from 1 to $\frac{3}{2}$. However, two such dislocations may be removed, with release of energy, by the reaction

$$a[100]+a[010]\rightarrow\tfrac{1}{2}a[111]+\tfrac{1}{2}a[11\bar{1}]. \quad . \quad . \quad . \quad . \quad (63)$$

5.314. *The Hexagonal Close Packed Structure.*—In the close packed hexagonal structure (and the structure derived from it by changing the ratio c/a) all atoms in the same basal plane are equivalent, but atoms in the planes above and below are not equivalent to these. The normal glide directions are therefore the close packed directions in the basal plane, with hexagonal indices $<11\bar{2}0>$. Suppose the stress is so oriented that the shear stress acting across the glide plane is small, which means that the hexagonal axis lies close to a principal axis of stress. Glide cannot occur on the base plane. The shortest remaining lattice displacement between equivalent atoms is in the direction of the hexagonal axis, from one atom to an atom in the next close packed plane but one, with Burgers vector $[000c]$. This is a principal axis of stress, and there is no component of shear stress tending to move this dislocation. Any other *complete* dislocation dissociates into dislocations of the types already considered. A hexagonal crystal pulled in, or perpendicular to, the basal plane, cannot deform by slip. It either fractures or twins by the motion of partial dislocations.

5.315. *The Sodium Chloride Structure.*—This structure has been discussed by Seitz (1950). The Burgers vector for slip is the line joining two neighbouring ions of like sign, $\frac{1}{2}a[110]$, and this defines the slip directions $<110>$.

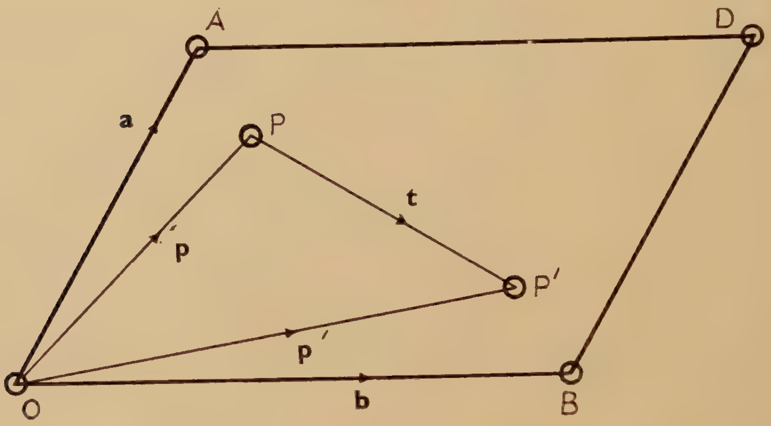
5.4. TWINNING AND PHASE TRANSFORMATIONS

It is obvious that suitable movements of partial dislocations can produce the shear displacements which occur in mechanical twinning and in some phase transformations in the solid state. Cottrell and Bilby (1951) have studied mechanical twinning in the body-centred and face-centred lattices from this standpoint, while Millard and Thompson (1952) and King (1952) have considered twinning in the hexagonal close packed structure. Frank (1952 b) has discussed the martensitic transformation from face centred to body centred iron; in this case the mechanism of the homogeneous shear is not discussed, but the inhomogeneous deformations which also occur are most naturally expressed in terms of screw dislocations. Stacking faults in cobalt and the relation between the cubic and the hexagonal close packed structures have also been studied (Christian 1951). A treatment of twinning from a rather different standpoint is due to Lifshits and Obreimov (1948). Before discussing the theories of Cottrell and Bilby, and of Lifshits and Obreimov, we make some general observations on the dislocations associated with twinning. We consider only twins in crystal lattices in which all atoms are equivalent, and neglect twins of the second kind in which the twinning plane is irrational.

5.41. *Twin Lattices, Twinning and Pole Dislocations*

Consider any lattice of atoms generated by three vectors which are not coplanar. Suppose (fig. 42) that $\mathbf{a}$ and $\mathbf{b}$ are the vectors joining any atom O to any other two atoms A and B. If these two vectors are not parallel, and the lines OA and OB do not pass through any other atoms, the vectors $\mathbf{a}$ and $\mathbf{b}$ are basic lattice vectors. They generate a two dimensional lattice, and the three dimensional lattice can be resolved into a stack of similar and similarly situated two dimensional lattices. The third basic vector $\mathbf{c}$ is the line joining O to any atom in the next sheet; we choose in particular that atom C whose projection normal to the *ab* plane is a point P lying within the two dimensional unit cell defined by OA and OB. If $\mathbf{h}$ is the normal distance between sheets, and $\mathbf{p}$ is the vector OP, the basic vectors are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{h}+\mathbf{p}$. There is one other point P′ in the two dimensional unit cell which is symmetrically related to P. By regarding D as the origin it is easily seen that the vectors $-\mathbf{a}$, $-\mathbf{b}$ and $\mathbf{h}+\mathbf{p}'-\mathbf{a}-\mathbf{b}$ define a lattice which is the mirror image of the first, the *twin lattice*. We may write this set of basic vectors more simply as $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{h}-\mathbf{p}$. (The mirror image may be regarded either as a mirror image in the plane OAB or as an image in the line perpendicular to this plane and passing through the mid-point of AB. For a lattice these are identical ; for a structure with basis and no centre of symmetry these processes yield different twins.) The coordinates of any atom in the twin lattice differ from those of a point in the original lattice by a multiple of the *twin vector* $\mathbf{t}=\mathbf{p}'-\mathbf{p}=-2\mathbf{p}$. The partial dislocation with Burgers vector $\mathbf{t}$ is a *twin dislocation.* If it lies in an *ab* plane of the original lattice, it is mobile, and may be called a *twinning dislocation.* As it moves across the plane, it leaves behind it a sheet of *stacking fault,* which may be regarded as a layer of twin one atom thick. For certain special choices of the vectors $\mathbf{a}$ and $\mathbf{b}$, this stacking fault has a low energy, so that the corresponding imperfections of the crystal are actually observed. A twinning vector $\mathbf{t}$ may, in a lattice of high symmetry, be common to two twinning planes, as occurs in the body-centred cubic structure.

Fig. 42



Twinning. O, A, B and D are lattice points, **a** and **b** lattice vectors in a plane. P is the normal projection of a point in a neighbouring plane. To the point P corresponds the symmetrical point P′, which is the normal projection of a point in the twin lattice.

To produce a twinned layer *m* sheets thick, twinning dislocations must pass across each of the *m* planes separating one sheet of atoms from the sheet below. Cottrell and Bilby pointed out that this could occur by the action of a modified form of the Frank–Read source. In the ordinary Frank–Read source, the dislocation in the glide plane rotates about a fixed *pole dislocation.* The pole dislocation may be the moving dislocation itself which has turned into another line and is there anchored, or there may be a dislocation node. In either case, all the Burgers vectors lie in the glide plane of the moving dislocation. If the Burgers vector of the pole dislocation has a component $\mathbf{n}$ normal to this glide plane, the stack of glide planes is distorted into a helical surface of pitch $\mathbf{n}$. Each time the moving dislocation rotates, it finds itself displaced a distance $\mathbf{n}$ normal to the glide plane. If $\mathbf{n}=\mathbf{h}$, the normal distance between planes of atoms, and the moving dislocation is not complete, but a twinning dislocation, a thick layer of twin can be formed.

We now consider the possibility of finding a suitable pole dislocation. In general, a dislocation of strength $\mathbf{h}$ cannot exist, for this would imply that a sheet of atoms originally lying distances $\mathbf{h}+\mathbf{p}$ from the atoms in the reference plane had been shifted to $2\mathbf{h}+\mathbf{p}$, while another sheet was interposed. The interposed sheet must lie either $\mathbf{h}+\mathbf{p}$ or $\mathbf{h}-\mathbf{p}$ from the atoms in the reference plane; the vector from an atom in the interposed sheet to an atom in the third sheet is then either $\mathbf{h}$ or $\mathbf{h}+2\mathbf{p}$. The vector $\mathbf{h}$ is not allowed, while $\mathbf{h}+2\mathbf{p}$ is allowed only if $3\mathbf{p}$ is a lattice vector, so that $\mathbf{h}+2\mathbf{p}=\mathbf{h}-\mathbf{p}$ is a vector of the twin lattice. This occurs for the (111) plane of the cubic close packed lattice. If the dislocation $\mathbf{h}$ does exist, it is sessile, but cannot function as a pole for twinning on the plane to which its Burgers vector is normal, or on other planes of the same type, for it lies in its own plane and does not thread it, while the component of its Burgers vector perpendicular to the planes of its own type which it threads is necessarily less than h, their spacing. Dislocations of strength $\mathbf{h}+\mathbf{p}$ or $\mathbf{h}-\mathbf{p}$ are possible, since the atoms then lie $2\mathbf{h}+2\mathbf{p}$ or $2\mathbf{h}$ from atoms in the reference plane, allowing the interposed atoms to lie at $\mathbf{h}+\mathbf{p}$ or $\mathbf{h}\pm\mathbf{p}$ from the atoms in the reference plane. However, $\mathbf{h}+\mathbf{p}$ and $\mathbf{h}-\mathbf{p}$ are lattice vectors of the original and the twin lattices respectively. The corresponding dislocations are therefore complete, and cannot be sessile. Their mobility may well interfere with the twinning process. Sessile dislocations of strength $2\mathbf{h}$ can always exist, because their atoms lie $3\mathbf{h}+\mathbf{p}$ from the atoms in the reference plane, which allows two interposed sheets to lie at $\mathbf{h}+\mathbf{p}$ and $2\mathbf{h}$, $\mathbf{h}-\mathbf{p}$ and $2\mathbf{h}$, or $\mathbf{h}+\mathbf{p}$ and $2\mathbf{h}+2\mathbf{p}$. The dislocation $2\mathbf{h}$ can be resolved into $\mathbf{h}+\mathbf{p}$ and $\mathbf{h}-\mathbf{p}$. The former is mobile in the original lattice, but the dissociation will not occur if the angle between $\mathbf{h}+\mathbf{p}$ and $\mathbf{h}-\mathbf{p}$ is obtuse, that is, if $h<p$. The line of the dislocation lies in the glide plane of the twinning dislocation $-2\mathbf{p}$, but if this plane is inclined at 60° to another plane of the same type, as happens in the body-centred cubic lattice, the component of its Burgers vector normal to this other plane is h, the interplanar spacing.

5.42. *Special Lattices*

5.421. *The Body-centred Cubic Lattice.*—Cottrell and Bilby consider a complete dislocation $\frac{1}{2}a[111]$ which lies in (112) but not along $[11\bar{1}]$. This dissociates without change of energy according to the scheme

$$\tfrac{1}{2}a[111]\rightarrow\tfrac{1}{3}a[112]+\tfrac{1}{6}a[11\bar{1}]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (64)$$

The first component is pure edge and sessile. It is of the type $2\mathbf{h}$, and its plane (112) is inclined at 60° to each of the equivalent planes $(\bar{1}21)$ and $(2\bar{1}1)$. It threads each of these, and its Burgers vector has a component normal to each of $h=a/\sqrt{6}$. The second component is a twinning dislocation in (112), and can rotate into the direction $[11\bar{1}]$, leaving a stacking fault in (112) behind it. In this orientation it is a pure screw. It is also a twinning dislocation in $(\bar{1}21)$ and $(2\bar{1}1)$, and under a suitable applied stress it starts to move in one of these. The requirements for progressive twinning are now satisfied.

5.422. *The Face-centred Cubic Lattice.*—Here Cottrell and Bilby state that the dissociation analogous to (64) is

$$\tfrac{1}{2}a[110] \rightarrow \tfrac{1}{3}a[111] + \tfrac{1}{6}a[11\bar{2}]. \quad \ldots \ldots \quad (65)$$

They show, by a detailed analysis, that progressive twinning cannot occur because the sessile dislocation $\tfrac{1}{3}a[111]$ lies in the glide plane of the twinning dislocation $\tfrac{1}{6}a[11\bar{2}]$. However, the analogy between (64) and (65) is incomplete, for $\tfrac{1}{3}a[111]$ is a sessile dislocation of type $\mathbf{h}$, and the true analogy is with the sessile dislocation $\tfrac{2}{3}a[111]$, which is of type $2\mathbf{h}$. Since the cosine of the angle between $\{111\}$ planes is $\tfrac{1}{3}$, this dislocation gives a lift of only $\tfrac{2}{3}h$ to twinning dislocations in another $\{111\}$ plane. A dislocation $a[111]$, which would give the necessary lift, would be both mobile and unstable. It seems that the absence of mechanical twinning in the face centred cubic lattice corresponding to the (111) twins produced during recrystallization is due to the lack of any suitable dislocation mechanism.

5.43. *The Analysis of Lifshits and Obreimov*

In experiments on calcite, Garber (1947) found that indentation with a knife edge produced a narrow twin which progressed gradually into the crystal under an increasing force, and reverted to the original lattice when the force was removed. When greater loads were applied, the twin progressed further, and did not disappear on reducing the load. Finally, the twin ran right across the crystal when a critical load was applied. Lifshits and Obreimov have analysed the process using a simple atomic model in which the restoring forces on an atom are linear in its displacement until a critical displacement is reached, after which the restoring force is proportional to the displacement from a second, twinned, position of equilibrium. They show that the Green's function for displacements may be expressed in terms of the lattice vibrations. The method of solution is to postulate a certain shape for the boundary of the twin, and verify that the relative displacements of atoms near the boundary still lie within the elastic range assumed. (Most of the calculations are, in fact, carried out for the case of a force applied to a line of atoms *within* the crystal.) They find that three cases are possible, according to the value assumed for certain parameters of the material: (1) All twins are elastic, and reach a finite depth under a finite load. On removing the load, they revert to the original lattice. (2) Twins penetrating to less than some critical depth are elastic, but those which penetrate further remain as twins when the load is removed. (3) At some value of the applied load the twin propagates into the crystal without further increase in the load. There may or may not be an elastic region before this.

REFERENCES

ANDRADE, E. N. DA C., and CHOW, Y. S., 1940, *Proc. Roy. Soc.* A, **175,** 290.

ANDRADE, E. N. DA C., and HENDERSON, C., 1951, *Phil. Trans.* A, **244,** 177.

BECKER, R., 1925, *Physik. Z.*, **26,** 919.

BILBY, B. A., 1950 a, *Proc. Phys. Soc.* A, **63,** 191 ; 1950 b, *Sheet Metal Industries*, p. 707 ; 1950 c, *J. Inst. Metals*, **76,** 613 ; 1951, *Research*, **4,** 387.

BLACKMAN, M., 1951, *Proc. Phys. Soc.* A, **64,** 942.

BOAS, W., and SCHMID, E., 1931, *Z. Physik*, **71,** 703.

BOGDANOFF, J. L., 1950, *J. Appl. Phys.*, **21,** 1258.

BRAGG, SIR W. L., 1940, *Proc. Phys. Soc.*, **52,** 54.

BRAGG, SIR W. L., and LOMER, W. M., 1949, *Proc. Roy. Soc.* A, **196,** 171.

BRAGG, Sir W. L., and NYE, J. F., 1947, *Proc. Roy. Soc.* A, **190,** 474.

BROWN, W. F., 1941, *Phys. Rev.*, **60,** 139.

BURGERS, J. M., 1939 a, *Proc. Acad. Sci. Amst.*, **42,** 293 ; 1939 b, *Ibid.*, **42,** 378.

BURTON, C. V., 1892, *Phil. Mag.* [5], **33,** 191.

BURTON, W. K., 1951, *Penguin Science News*, **21,** 26.

BURTON, W. K., CABRERA, N., and FRANK, F. C., 1951, *Phil. Trans.* A, **243,** 299.

CAHN, R. W., 1948, *Discussion to Burgers, W. G.*, 1947 *Bristol Conference on Strength of Solids* (London : The Physical Society), p. 134 ; 1949, *J. Inst. Metals*, **76,** 121 ; 1950, *Progress in Metal Physics*, **2** (ed. Chalmers) (London : Butterworth), p. 151.

CHALMERS, B., and MARTIUS, URSULA, 1951, *Nature, Lond.*, **167,** 681.

CHRISTIAN, J. W., 1951, *Proc. Roy. Soc.* A, **206,** 51.

COLONNETTI, G., 1915, *R.C. Accad. Lincei* [5], **24,** 404.

COTTRELL, A. H., 1948, 1947 *Bristol Conference on Strength of Solids* (London : The Physical Society), p. 30 ; 1949, *Progress in Metal Physics*, **1** (ed. Chalmers) (London : Butterworth), p. 77 ; 1952 a, *Report of Ninth Solvay Conference* (Brussels : Stoops), p. 487 ; 1952 b, *Ibid.*, p. 421 ; 1952 c, *Phil. Mag.*, **43,** 645.

COTTRELL, A. H., and BILBY, B. A., 1949, *Proc. Phys. Soc.* A, **62,** 49 ; 1951, *Phil. Mag.*, **42,** 573.

COTTRELL, A. H., and JASWON, M. A., 1949, *Proc. Roy. Soc.* A, **199,** 104.

CRUSSARD, C., 1945, *Rev. Métall.*, **42,** 329 ; 1947, *Metal Treatment*, 149 ; 1950, *Métaux et Corrosion*, **25,** 203.

DAWSON, I. M., and VAND, V., 1951, *Proc. Roy. Soc.* A, **206,** 555.

DEAN, W. R., and MANN, E. H., 1949, *Proc. Camb. Phil. Soc.*, **45,** 131.

DEAN, W. R., and WILSON, A. H., 1947, *Proc. Camb. Phil. Soc.*, **43,** 205.

DEHLINGER, U., 1929, *Ann. Phys. Lpz.* [5], **2,** 749.

DEHLINGER, U., and KOCHENDÖRFER, A., 1940, *Z. Physik*, **116,** 576.

DEXTER, D. L., 1952, *Phys. Rev.*, **85,** 936.

DIETZE, H.-D., 1952 a, *Z. Physik*, **131,** 156 ; 1952 b, *Ibid.*, **132,** 107.

DUNN, C. G., and DANIELS, F. W., 1951, *Trans. Amer. Inst. Min. Met. Eng.*, **191,** 147.

DUNN, C. G., DANIELS, F. W., and BOLTON, M. J., 1950, *Trans. Amer. Inst. Min. Met. Eng.*, **188,** 368.

DUNN, C. G., and LIONETTI, F., 1949, *Trans. Amer. Inst. Min. Met. Eng.*, **185,** 125.

ESHELBY, J. D., 1949 a, *Phil. Mag.*, **40,** 903 ; 1949 b, *Proc. Phys. Soc.* A, **62,** 307 ; 1951, *Phil. Trans.* A, **244,** 87.

ESHELBY, J. D., FRANK, F. C., and NABARRO, F. R. N., 1951, *Phil. Mag.*, **42,** 351.

ESHELBY, J. D., and STROH, A. N., 1951, *Phil. Mag.*, **42,** 1401.

FENN, R. W., HIBBARD, W. R., and LEPPER, H. A., 1950, *Trans. Amer. Inst. Min. Met. Eng.*, **188,** 175.

FILON, L. N. G., 1921, *Brit. Assoc. Adv. Sci. Report*, **89,** 305.

Fisher, J. C., and Dunn, C. G., 1952, *Imperfections in Nearly Perfect Crystals* (ed. Shockley) (New York: Wiley).

Foreman, A. J., Jaswon, M. A., and Wood, J. K., 1951, *Proc. Phys. Soc.* A, **64,** 156.

Frank, F. C., 1948, 1947 *Bristol Conference on Strength of Solids* (London: The Physical Society), p. 46; 1949 a, *Research*, **2,** 542; 1949 b, *Proc. Phys. Soc.* A, **62,** 131; 1949 c, *Ibid.*, **62,** 202; 1949 d, *Physica, 's Grav.*, **15,** 131; 1950 a, *Plastic Deformation of Crystalline Solids, Pittsburgh, Carnegie-Institute of Technology and U.S. Office of Naval Research*, p. 89; 1950 b, *Ibid.*, p. 150; 1951 a, *Phil. Mag.*, **42,** 809; 1951 b, *Acta. Cryst.*, **4,** 497; 1951 c, *Proc. Phys. Soc.* A, **64,** 941; 1952 a, *Adv. Phys.*, **1,** 91; 1952 b (private communication).

Frank, F. C., and Read, W. T., 1950, *Phys. Rev.*, **79,** 722.

Frank, F. C., and van der Merwe, J. H., 1949 a, *Proc. Roy. Soc.* A, **198,** 205; 1949 b, *Ibid.*, **198,** 216; 1949 c, *Ibid.*, **200,** 125.

Frenkel, J., and Kontorova, T., 1938, *Phys. Z. Sowjetunion*, **13,** 1.

Garber, R. I., 1947, *J. Physics, U.S.S.R.*, **11,** 55.

Goodier, J. N., 1939, *Proc. 5th Internat. Congress. Appl. Mech.* (1938) (New York: Wiley), p. 129.

Guinier, A., and Tennevin, J., 1948, *C.R. Acad. Sci. Paris*, **226,** 1530; 1950, *Progress in Metal Physics*, **2** (ed. Chalmers) (London: Butterworth), p. 177.

Heidenreich, R. D., and Shockley, W., 1948, 1947 *Bristol Conference on Strength of Solids* (London: The Physical Society), p. 57.

Jaswon, M. A., and Foreman, A. J. E., 1952, *Phil. Mag.*, **43,** 201.

Jillson, D. C., 1950, *Trans. Amer. Inst. Min. Met. Eng.*, **188,** 1009.

King, R., 1952 (private communication).

Kochendörfer, A., and Seeger, A., 1950, *Z. Physik*, **127,** 533.

Koehler, J. S., 1941, *Phys. Rev.*, **60,** 397; 1949, *Ibid.*, **75,** 106; 1952, *Ibid.*, **85,** 480.

Lacombe, P., 1948, 1947 *Bristol Conference on Strength of Solids* (London: The Physical Society), p. 91.

Lacombe, P., and Beaujard, L., 1947, *J. Inst. Met.*, **74,** 1.

Larmor, Sir J., 1897, *Phil. Trans.* A, **190,** 205; 1900, *Aether and Matter* (Cambridge: University Press), p. 335.

László, F., 1950, *J. Iron Steel Inst.*, 1950, No. I, 5.

Leibfried, G., 1949, *Z. Physik*, **126,** 781; 1950, *Ibid.*, **127,** 344; 1951, *Ibid.*, **130,** 214.

Leibfried, G., and Dietze, H.-D., 1949, *Z. Physik*, **126,** 790; 1951, *Ibid.*, **131,** 113.

Leibfried, G., and Lücke, K., 1949, *Z. Physik*, **126,** 450.

Lifshits, I. M., and Obreimov, I. V., 1948, *Izv. Akad. Nauk. Ser. Phys.*, **12,** 65 (A.E.R.E. Harwell, Translation 11/3/5/154, U.D.C., 548. 122).

Lomer, W. M., 1949, *Proc. Roy. Soc.* A, **196,** 182; 1951, *Phil. Mag.*, **42,** 1327; 1952 (private communication).

Lomer, W. M., and Nye, J. F., 1952, *Proc. Roy. Soc.* A, **212,** 576.

Love, A. E. H., 1927, *The Mathematical Theory of Elasticity* (Cambridge: University Press).

Lucas, L. N. D., 1951, *Proc. Phys. Soc.* A, **64,** 943.

Mackenzie, J. K., and Sondheimer, E. H., 1950, *Phys. Rev.*, **77,** 264.

Maddin, R., Mathewson, C. H., and Hibbard, W. R., 1948, *Trans. Amer. Inst. Min. Met. Eng.*, **175,** 86.

Mann, Elizabeth H., 1949, *Proc. Roy. Soc.* A, **199,** 376.

Masing, G., 1944, *Nachr. Akad. Wiss. Göttingen* (*Math.-Phys. Klasse*), 81; 1948, *Z. Physik*, **124,** 586.

Masing, G., and Polanyi, M., 1923, *Ergeb. exact. Naturwiss.*, **2,** 177.

Mathewson, C. H., 1944, *Trans. Amer. Soc. Metals*, **32,** 38 ; 1951, *Trans. Connecticut Acad. Art. Sci.*, **38,** 213.

Michell, J. H., 1899, *Proc. Lond. Math. Soc.*, **31,** 100.

Millington, W. E. W., and Thompson, F. C., 1924, *J. Inst. Metals*, **31,** 81.

Mott, N. F., 1951, *Proc. Phys. Soc.* B, **64,** 729 ; 1952 a (private communication) ; 1952 b, *Imperfections in Nearly Perfect Crystals* (ed. Shockley) (New York : Wiley).

Mott, N. F., and Nabarro, F. R. N., 1948, 1947 *Bristol Conference on Strength of Solids* (London : The Physical Society), p. 1.

Nabarro, F. R. N., 1947, *Proc. Phys. Soc.*, **59,** 256 ; 1948 a, 1947 *Bristol Conference on Strength of Solids* (London : The Physical Society), p. 38 ; 1948 b, *Ibid.*, p. 75 ; 1950, *Phys. Rev.*, **79,** 894 ; 1951 a, *Phil. Mag.*, **42,** 213 ; 1951 b, *Ibid.*, **42,** 1224 ; 1951 c, *Proc. Roy. Soc.* A, **209,** 278.

Neményi, P., 1931, *Z. angew. Math. Mech.*, **11,** 59.

Nye, J. F., 1949, *Proc. Roy. Soc.* A, **200,** 47.

Orowan, E., 1934 a, *Z. Physik*, **89,** 605 ; 1934 b, *Ibid.*, **89,** 634 ; 1948, *Internal Stresses in Metals* (London : The Institute of Metals), p. 451.

Paneth, H. R., 1950, *Phys. Rev.*, **80,** 708.

Peach, M. O., 1951, *J. Appl. Phys.*, **22,** 1359.

Peach, M. O., and Koehler, J. S., 1950, *Phys. Rev.*, **80,** 436.

Peierls, R. E., 1940, *Proc. Phys. Soc.*, **52,** 23.

Polanyi, M., 1934, *Z. Phys.*, **89,** 660.

Prandtl, L., 1928, *Z. angew. Math. Mech.*, **8,** 85.

Read, W. T., 1952 (private communication).

Read, W. T., and Shockley, W., 1950, *Phys. Rev.*, **78,** 275 ; 1952, *Imperfections in Nearly Perfect Crystals* (ed. Shockley) (New York : Wiley).

Schmid, E., 1924, *Proc. Internat. Cong. Appl. Mech., Delft*, p. 342.

Seeger, A., 1951, *Naturwiss.*, **38,** 526.

Seeger, A., and Kochendörfer, A., 1951, *Z. Physik*, **130,** 321.

Segedin, C. M., 1951, *Proc. Camb. Phil. Soc.*, **47,** 396.

Seitz, F., 1950 a, *Plastic Deformation of Crystalline Solids, Pittsburgh, Carnegie Institute of Technology and U.S. Office of Naval Research*, p. 1 ; 1950 b, *Phys. Rev.*, **79,** 890, 1002, 1003 ; 1950 c, *Phys. Rev.*, **79,** 723 ; 1950 d, *Plastic Deformation of Crystalline Solids, Pittsburgh, Carnegie Institute of Technology and U.S. Office of Naval Research*, p. 37 ; 1952, *Adv. Phys.*, **1,** 43.

Shockley, W., 1947 (private communication), quoted by Mott and Nabarro (1948) ; 1952, *Report of Ninth Solvay Conference* (Brussels : Stoops), p. 431.

Shockley, W., and Read, W. T., 1949, *Phys. Rev.*, **75,** 692.

Smakula, A., and Klein, M. W., 1949, *J. Opt. Soc. America*, **39,** 445, 890.

Smith, C. L., 1948, *Proc. Phys. Soc.*, **61,** 201.

Smollett, M., and Blackman, M., 1951, *Proc. Phys. Soc.* A, **64,** 683.

Starr, A. T., 1928, *Proc. Camb. Phil. Soc.*, **24,** 489.

Stieltjes, T. J., 1885, *Acta Math.*, **6,** 321.

Sutton, H., and Willstrop, J. W., 1937, discussion to Gayler, M.L.V., *J. Inst. Metals*, **60,** 249.

Taylor, Sir G. I., 1934, *Proc. Roy. Soc.* A, **145,** 362.

Taylor, Sir G. I., and Elam, C. F., 1923, *Proc. Roy. Soc.* A, **102,** 643 ; 1926, *Ibid.*, **112,** 337.

Thompson, N., 1952 (private communication).

Thompson, N., and Millard, D. J., *Phil. Mag.*, **43,** 421.

van der Merwe, J. H., 1949, *Discuss. Faraday Soc.*, **5,** 201 ; 1950, *Proc. Phys. Soc.* A, **63,** 616.

van der Merwe, J. H., and Frank, F. C., 1949, *Proc. Phys. Soc.* A, **62,** 315.

Verma, A. R., 1951, *Phil. Mag.*, **42,** 1005.

Volterra, V., 1907, *Ann. Ecole Norm. Supér.* [3], **24,** 400.

Weingarten, G., 1901, *R.C. Accad. Lincei* [5], **10,** 57.

Wilman, H., 1950, *Nature, Lond.*, **165,** 321 ; 1951, *Proc. Phys. Soc.* A, **64,** 329.

Wilson, A. J. C., 1949, *Research,* **2,** 541.

Wyatt, O. H., 1950, *Thesis,* Cambridge.

Zener, C., 1948 a, *Fracturing of Metals, Amer. Soc. Metals, Cleveland, Ohio,* p. 3 ; 1948 b, *Elasticity and Anelasticity of Metals* (Chicago : The University Press), Ch. IV.

We beg to acknowledge the source of the following text-figures :

Figure 4 from Orowan, 1934, *Z. Physik,* **89,** 634.

Figure 5 from Polanyi, 1934, *Z. Physik,* **89,** 660.

Figure 6 from Taylor, 1934, *Proc. Roy. Soc.* A, **145,** 362.

Figure 7 from Mott and Nabarro, *Physical Society* 1947 *Conference Report,* p. 1.

Figure 8 modified from fig. 2 of Mott and Nabarro.

Figures 9 and 10 from Burgers, 1939, *Proc. Acad. Sci. Amsterdam,* **42,** 293.

Figure 17 from Frank and Read, 1950, *Phys. Rev.,* **79,** 722.

Figures 24, 24 (*b*) after W. T. Read, Courtesy of Bell Telephone Laboratories, Inc.

Figure 27 from Read and Shockley, 1950, *Phys. Rev.,* **78,** 275.

Figure 31 from Frank, *Plastic Deformation of Crystalline Solids* (U.S. Office of Naval Research) (Pittsburgh Conference, May, 1950).

Figure 34 from Nabarro, 1947, *Proc. Phys. Soc.,* **59,** 256.

Figure 35 from van der Merwe, 1950, *Proc. Phys. Soc.* A, **63,** 616.

Figure 37 from Frank and van der Merwe, 1949, *Proc. Roy. Soc.* A, **198,** 216.

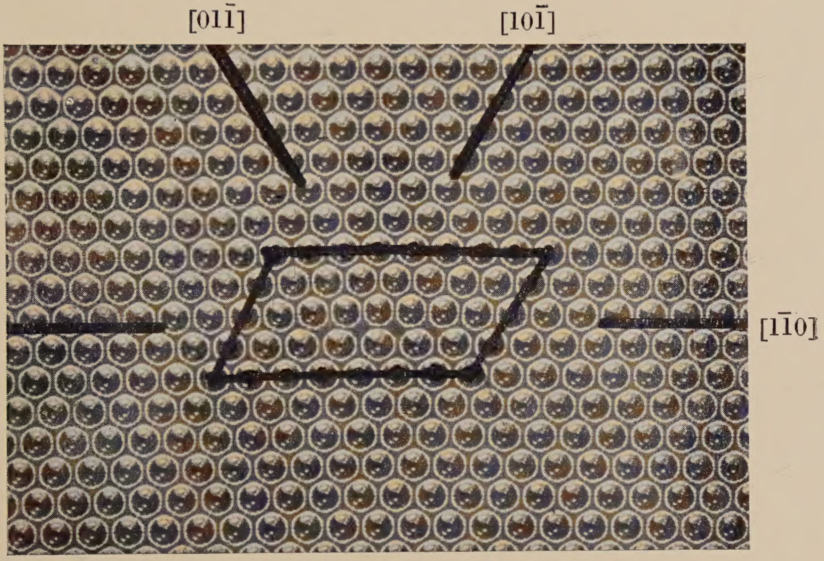
Figure 38 from Frank and van der Merwe, 1949, *Proc. Roy. Soc.* A, **200,** 125.

Figure 40 from Mott and Nabarro (*loc. cit.*).

ADDENDUM

'Properties of Helium Three at Low Temperatures', J. G. Daunt, *Advances in Physics*, **1,** 209, 1952.

In §6(*c*) of the above article reference was unfortunately omitted to a paper by L. Goldstein and M. Goldstein (*J. Chem. Phys.*, **18,** 538, 1950) which considers theoretically the magnetic properties of liquid ^{3}He. In this paper the exchange energy originating in the interatomic mutual potential energy of the ^{3}He atoms (both Slater-Kirkwood and Margenau potentials are considered) is computed and is compared with the kinetic energy of the system. It is concluded, as reported in §6(*c*), that nuclear ferromagnetism in liquid ^{3}He is not a stable configuration.



A dislocation in a raft of bubbles (photograph by W. M. Lomer). The glide direction $[1\bar{1}0]$, the rows of atoms inserted along $[10\bar{1}]$ and $[01\bar{1}]$, and a Burgers circuit encircling the dislocation, are indicated.

Annals of Science

A QUARTERLY REVIEW OF THE HISTORY OF SCIENCE SINCE THE RENAISSANCE

EDITORS

D. McKIE, D.Sc., Ph.D.,
University College, London.

HARCOURT BROWN, M.A., Ph.D.,
Brown University, Providence, R.I., U.S.A.

H. W. ROBINSON,
Former Librarian,
Royal Society of London.

N. H. de V. HEATHCOTE, B.Sc., Ph.D.,
University College, London.

ANNUAL SUBSCRIPTION
£2 0s. 0d.
OR
10s. 6d.
PER PART
POST FREE

ALERE FLAMMAM

Contents of Vol. 8, No. 1, March 1952

THE EARLY OBSERVATORY INSTRUMENTS OF TRINITY COLLEGE, CAMBRIDGE. BY DEREK J. PRICE, PH.D.

JOHN FRANCIS VIGANI. BY L. J. M. COLEBY, M.A., M.SC., PH.D.

CHEMISTRY IN ROZIER'S JOURNAL.—VIII AND IX. BY E. W. J. NEAVE, M.C., M.SC., PH.D.

BOOLE'S PHILOSOPHY OF LOGIC. BY MARY B. HESSE, M.SC., PH.D.

SIR CHARLES CAVENDISH AND HIS LEARNED FRIENDS. BY JEAN JACQUOT.

THE TEACHING OF SCIENCE AT OXFORD IN THE NINETEENTH CENTURY. BY F. SHERWOOD TAYLOR, M.A., B.SC., PH.D.

YLOR & FRANCIS, LTD., Red Lion Court, Fleet Street, LONDON, E.C.4

FILED IN STACKS

Another Taylor & Francis Publication

Atomic Scientists' News

Journal of the Atomic Scientists' Association

President . . . . Professor M. H. L. Pryce, F.R.S.
Executive Vice-President . . . . . . Professor Kathleen Lonsdale, F.R.S.

Vice-Presidents:
Sir Wallace Akers.
Professor The Rt. Hon. Lord Cherwell, P.C., F.R.S.
Professor H. S. W. Massey, F.R.S.
Professor N. F. Mott, F.R.S.
Professor R. E. Peierls, F.R.S.
Professor F. A. Paneth, F.R.S.
Professor H. W. B. Skinner, F.R.S.
Professor P. M. S. Blackett, F.R.S.
Sir John Cockcroft, F.R.S.
Sir Charles Darwin, F.R.S.
Professor P. B. Moon, F.R.S.
Professor M. L. E. Oliphant, F.R.S.
Professor F. E. Simon, F.R.S.
Professor Sir Geoffrey Taylor, F.R.S.
Professor Sir George Thomson, F.R.S.
Professor C. F. Powell, F.R.S.

General Sec.: Dr. J. L. Michiels.
Treasurer: Dr. L. E. J. Roberts.

- The Atomic Scientists' Association is an association of scientists whose work has given them special knowledge of the consequences for the world of the use and misuse of atomic energy. To make known the true facts about atomic energy and its implications, it publishes every two months the **Atomic Scientists' News.**
- Full membership of the Association is open to all scientists able to put before the public an informed opinion upon some aspect of atomic energy. Others interested in its work may become associate members. Enquiries c/o Professor J. Rotblat, Physics Department, St. Bartholomew's Hospital, London, E.C.1.

Vol. 1 **JULY 1952** **No. 6**

CONTENTS

Editorial
The Highways and Byways of Science - - - - Prof. L. Infeld
Comments on Universities in the United States - Prof. S. K. Allison
Applications of Atomic Energy in Plant Science - Dr. W. D. Allen
The Chemistry of Plutonium and Americium
Nuclear Power
Review of the Month - - - - - - F. R. N. Nabarro

Price to non-members 6/- (*plus postage*) **Annual subscription £1 12s. 6d.** (*post free*)

Non-members apply to the Printers and Publishers:—

Messrs. TAYLOR & FRANCIS, LTD.
Red Lion Court, Fleet Street, London, E.C.4